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THE
DOCTRINE
OF
CHANCES:

OR,
A Method of Calculating the Probability
of Events in Play.



By *A. De Moivre*. F. R. S.

L O N D O N:

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THE
DOCTRINE
OF
CHANGES



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T O

*Sir Isaac Newton, Kt. President
of the Royal Society.*

S I R

TH E greatest help I have received in writing upon this Subject having been from your Incomparable Works, especially your Method of Series; I think it my Duty publickly to acknowledge, that the Improvements I have made in the matter here treated of, are principally derived from your self. The great benefit which has accrued to me in this respect, requires my share in the general Tribute of Thanks due to you from the Learned World: But one advantage, which is more particularly my own, is the Honour I have frequently had of being admitted to your private Conversation, wherein the doubts I have had upon any Subject relating to *Mathematics*, have been resolved by you with the greatest Humanity and Condescension. Those
Marks

The Dedication.

Marks of your Favour are the more valuable to me, because I had no other pretence to them, but the earnest desire of understanding your sublime and universally useful Speculations. I should think my self very happy, if, having given my Readers a Method of calculating the Effects of Chance, as they are the result of Play, and thereby fix'd certain Rules, for estimating how far some sort of Events may rather be owing to Design than Chance, I could by this small Essay excite in others a desire of prosecuting these Studies, and of learning from your Philosophy how to collect, by a just Calculation, the Evidences of exquisite Wisdom and Design, which appear in the *Phænomena* of Nature throughout the Universe. I am, with the utmost Respect,

Sir,

Your most Humble,

and Obedient Servant,

A. De Moivre.



P R E F A C E.

TIS now about Seven Years, since I gave a Specimen in the Philosophical Transactions, of what I now more largely treat of in this Book. The occasion of my then undertaking this Subject was chiefly owing to the Desire and Encouragement of the Honourable Mr. Francis Robartes, who, upon occasion of a French Treatise, called, *L'Analyse des jeux de Hazard*, which had lately been Published, was pleased to propose to me some Problems of much greater difficulty than any he had found in that Book; which having solved to his Satisfaction, he engaged me to Methodise those Problems, and to lay down the Rules which had led me to their Solution. After I had proceeded thus far, it was enjoined me by the Royal Society, to communicate to them what I had discovered on this Subject, and thereupon it was ordered to be published in the Transactions, not as a matter relating only to Play, but as containing some general Speculations not unworthy to be considered by the Lovers of Truth.

I had not at that time read any thing concerning this Subject, but Mr. Huygens's Book, *de Ratiociniis in Ludo Alex*, and a little English Piece (which was properly a translation of it) done by a very ingenious Gentleman, who, tho' capable of carrying the matter a great deal farther, was contented to follow his Original; adding only to it the computation of the Advantage of the Setter in the Play called Hazard; and some few things more. As for the French Book, I had run it over but cursorily, by reason I had observed that the Author chiefly insisted on the Method of Huygens, which I was absolutely re-

solved to reject, as not seeming to me to be the genuine and natural way of coming at the Solution of Problems of this kind. However, had I allowed my self a little more time to consider it, I had certainly done the Justice to its Author, to have owned that he had not only illustrated Huygens's Method by a great variety of well chosen Examples, but that he had added to it several curious things of his own Invention.

Tho' I have not followed Mr. Huygens in his Method of Solution, 'tis with very great pleasure that I acknowledge the Obligations I have to him; his Book having Settled in my Mind the first Notions of this Doctrine, and taught me to argue about it with certainty.

I had said in my Specimen, that Mr. Huygens was the first who had Published the Rules of this Calculation, intending thereby to do justice to that great Man; but what I then said was misinterpreted, as if I had designed to wrong some Persons who had considered this matter before him, and a passage was cited against me out of Huygens's Preface, in which he saith, Sciendum vero quod jam pridem, inter Præstantissimos totâ Galliâ Geometras, Calculus hic fuerit agitatus; ne quis indebitam mihi primæ Inventionis gloriam hac in re tribuat. But what follows immediately after, had it been minded, might have cleared me from any Suspicion of injustice. The words are these Cæterum illi difficillimis quibusque Quæstionibus se invicem exercere Soliti, methodum Suam quisque occultam retinuerunt, adeo ut a primis elementis hanc materiam evolvere mihi necesse fuerit. By which it appears, that tho' Mr. Huygens was not the first who had applied himself to those sorts of Questions, he was nevertheless the first who had published Rules for their Solution; which is all that I asserted.

Since the printing of my Specimen, Mr. de Monmort, Author of the Analyse des jeux de Hazard, Published a Second Edition of that Book, in which he has particularly given many proofs of his singular Genius, and extraordinary Capacity; which Testimony I give both to Truth, and to the Friendship with which he is pleased to Honour me.

Such a Treatise as this is may be useful to several ends; the first of which is, that there being in the World several inquisitive Persons, who are desirous to know what foundation they

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go upon, when they engage in Play, whether from a motive of Gain, or barely Diversion, they may, by the help of this or the like Tract, gratifie their curiosity, either by taking the pains to understand what is here Demonstrated, or else making use of the conclusions, and taking it for granted that the Demonstrations are right.

Another use to be made of this Doctrine of Chances is, that it may serve in Conjunction with the other parts of the Mathematicks, as a fit introduction to the Art of Reasoning; it being known by experience that nothing can contribute more to the attaining of that Art, than the consideration of a long Train of Consequences, rightly deduced from undoubted Principles, of which this Book affords many Examples. To this may be added, that some of the Problems about Chance having a great appearance of Simplicity, the Mind is easily drawn into a belief, that their Solution may be attained by the meer Strength of natural good Sense; which generally proving otherwise, and the Mistakes occasioned thereby being not unfrequent, 'tis presumed that a Book of this Kind, which teaches to distinguish Truth from what seems so nearly to resemble it, will be look'd upon as a help to good Reasoning.

Among the several Mistakes that are committed about Chance, one of the most common and least suspected, is that which relates to Lotterys. Thus, supposing a Lottery wherein the proportion of the Blanks to the Prizes is as five to one; 'tis very natural to conclude that therefore five Tickets are requisite for the Chance of a Prize; and yet it may be proved Demonstratively, that four Tickets are more then sufficient for that purpose, which will be confirmed by often repeated Experience. In the like manner, supposing a Lottery wherein the proportion of the Blanks to the Prizes is as thirty nine to One, (such as was the Lottery of 1710) it may be proved, that in twenty eight Tickets, a Prize is as likely to be taken as not; which tho' it may seem to contradict the common Notions, is nevertheless grounded upon infallible Demonstration.

When the Play of the Royal Oak was in use, some Persons who lost considerably by it, had their Losses chiefly occasioned by an Argument of which they could not perceive the Fallacy. The Odds against any particular Point of the Ball were one and Thirty to One, which intituled the Adventurers, in case they

were winners, to have thirty two Stakes returned, including their own; instead of which they having but eight and Twenty, it was very plain that on the Single account of the disadvantage of the Play, they lost one eighth part of all the Money they play'd for. But the Master of the Ball maintained that they had no reason to complain; since he would undertake that any particular point of the Ball should come up in two and Twenty Throws; of this he would offer to lay a Wager, and actually laid it when required. The seeming contradiction between the Odds of one and thirty to One, and Twenty two Throws for any Chance to come up, so perplexed the Adventurers, that they began to think the Advantage was on their side; for which reason they play'd on and continued to lose.

The Doctrine of Chances may likewise be a help to cure a Kind of Superstition, which has been of long standing in the World, viz. that there is in Play such a thing as Luck, good or bad. I own there are a great many judicious people, who without any other Assistance than that of their own reason, are satisfied, that the Notion of Luck is meerly Chimerical; yet I conceive that the ground they have to look upon it as such, may still be farther enforced from some of the following Considerations.

If by saying that a Man has had good Luck, nothing more was meant than that he has been generally a Gainer at play, the Expression might be allowed as very proper in a short way of speaking: But if the Word good Luck be understood to signify a certain predominant quality, so inherent in a Man, that he must win whenever he Plays, or at least win oftner than lose, it may be denied that there is any such thing in nature.

The Asserters of Luck are very sure from their own Experience, that at some times they have been very Lucky, and that at other times they have had a prodigious run of ill Luck against them, which whilst it continued obliged them to be very cautious in engaging with the fortunate; but how Chance should produce those extraordinary Events, is what they cannot conceive; They would be glad for Instance to be Satisfied, how they could lose Fifteen Games together at Piquet, if ill Luck had not strangely prevailed against them. But if they will be pleased to consider the Rules delivered in this Book, they will see that tho' the Odds against their losing so many times together be very great

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great, viz. 32767 to 1, yet that the Possibility of it is not destroy'd by the greatness of the Odds, there being One Chance in 32768 that it may so happen, from whence it follows, that it was still possible to come to pass without the Intervention of what they call Ill Luck.

Besides, This Accident of losing Fifteen times together at Piquet, is no more to be imputed to ill Luck, than the Winning with one single Ticket the Highest Prize, in a Lottery of 32768 Tickets, is to be imputed to good Luck, since the Chances in both Cases are perfectly equal. But if it be said that Luck has been concerned in this latter Case, the Answer will be easy; for let us suppose Luck not existing, or at least let us suppose its Influence to be suspended, yet the Highest Prize must fall into some Hand or other, not by Luck, (for by the Hypothesis that has been laid aside) but from the meer Necessity of its falling somewhere.

Those who contend for Luck, may, if they please, alledge other Cases at Play, much more unlikely to happen than the Winning or Losing Fifteen Games together, yet still their Opinion will never receive any Addition of Strength from such Suppositions: For, by the Rules of Chance, a time may be computed, in which those Cases may as probably happen as not; nay, not only so, but a time may be computed, in which there may be any proportion of Odds for their so happening.

But supposing that Gain and Loss were so fluctuating, as always to be distributed equally, whereby Luck would certainly be annihilated; would it be reasonable in this Case to attribute the Events of Play to Chance alone? I think, on the contrary, it would be quite otherwise, for then there would be more reason to suspect that some unaccountable Fatality did Rule in it: Thus, If two Persons play at Cross and Pile, and Chance alone be suppos'd to be concern'd in regulating the fall of the Piece, is it probable that there should be an Equality of Heads and Crosses? It is Five to Three that in four times there will be an inequality; 'tis Eleven to Five in six, 93 to 35 in Eight, and about 12 to 1 in a hundred times: Wherefore Chance alone by its Nature constitutes the Inequalities of Play, and there is no need to have recourse to Luck to explain them.

Further, The same Arguments which explode the Notion of Luck, may, on the other side, be useful in some Cases to establish a due comparison between Chance and Design: We may imagine Chance and Design to be as it were in Competition with each other, for the pro-
duction

duction of some sorts of Events, and may calculate what Probability there is, that those Events should be rather owing to one than to the other. To give a familiar Instance of this, Let us suppose that two Packs of Piquet-Cards being sent for, it should be perceived that there is, from Top to Bottom, the same Disposition of the Cards in both Packs; Let us likewise suppose that, some doubt arising about this Disposition of the Cards, it should be questioned whether it ought to be attributed to Chance, or to the Maker's Design: In this case the Doctrine of Combination decides the Question, since it may be proved by its Rules, that there are the Odds of above 26313083 Millions of Millions of Millions of Millions to One, that the Cards were designedly set in the Order in which they were found.

From this last Consideration we may learn, in many Cases, how to distinguish the Events which are the effect of Chance, from those which are produc'd by Design: The very Doctrine that finds Chance where it really is, being able to prove by a gradual Increase of Probability, till it arrive at Demonstration, that where Uniformity, Order and Constancy reside, there also reside Choice and Design.

Lastly, One of the Principal Uses to which this Doctrine of Chances may be apply'd, is the discovering of some Truths, which cannot fail of pleasing the Mind, by their Generality and Simplicity; the Admirable Connexion of its Consequences will increase the Pleasure of the Discovery; and the seeming Paradoxes wherewith it abounds, will afford very great matter of Surprise and Entertainment to the Inquisitive. A very remarkable Instance of this nature may be seen in the prodigious Advantage which the repetition of Odds will amount to; Thus, Supposing I play with an Adversary who allows me the Odds of 43 to 40, and agrees with me to play till 100 Stakes are won or lost on either side, on condition that I give him an Equivalent for the Gain I am intitled to by the Advantage of my Odds; the Question is what Equivalent I am to give him, on supposition we play a Guinea a Stake: The Answer is 99 Guineas and above 18 Shillings, which will seem almost incredible, considering the smallness of the Odds of 43 to 40. Now let the Odds be in any Proportion given, and let the Number of Stakes to be played for be never so great, yet one General Conclusion will include all the possible Cases, and the application of it to Numbers may be wrought in less than a Minutes time.

I have explain'd, in my Introduction to the following Treatise, the chief Rules on which the whole Art of Chances depends; I have done

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done it in the plainest manner that I could think of, to the end it might be (as much as possible) of General Use. I flatter my self that those who are acquainted with Arithmetical Operations, will, by the help of the Introduction alone, be able to solve a great Variety of Questions depending on Chance: I wish, for the Sake of some Gentlemen who have been pleased to subscribe to the printing of my Book, that I could every where have been as plain as in the Introduction; but this was hardly practicable, the Invention of the greatest part of the Rules being intirely owing to Algebra; yet I have, as much as possible, endeavour'd to deduce from the Algebraical Calculation several practical Rules, the Truth of which may be depended upon, and which may be very useful to those who have contented themselves to learn only common Arithmetick.

'Tis for the Sake of those Gentlemen that I have enlarged my first Design, which was to have laid down such Precepts only as might be sufficient to deduce the Solution of any difficult Problem relating to my Subject: And for this reason I have (towards the latter end of the Book) given the Solution, in Words at length, of some easy Problems, which might else have been made Corollaries or Consequences of the Rules before deliver'd: The single Difficulty which may occur from Pag. 155 to the end, being only an Algebraical Calculation belonging to the 49th Problem, to explain which fully would have required too much room.

On this Occasion, I must take notice to such of my Readers as are well vers'd in Vulgar Arithmetick, that it would not be difficult for them to make themselves Masters, not only of all the Practical Rules in this Book, but also of more useful Discoveries, if they would take the small Pains of being acquainted with the bare Notation of Algebra, which might be done in the hundredth part of the Time that is spent in learning to read Short-hand.

One of the Principal Methods I have made use of in the following Treatise, has been the Doctrine of Combinations, taken in a Sence somewhat more extensive, than as it is commonly understood. The Notion of Combinations being so well fitted to the Calculation of Chance, that it naturally enters the Mind whenever any Attempt is made towards the Solution of any Problem of that kind. It was this which led me in course to the Consideration of the Degrees of Skill in the Adventurers at Play, and I have made use of it in most parts of this Book, as one of the Data that enter the Question; it being so far from perplexing the Calculation, that on
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the contrary it is rather a Help and an Ornament to it: It is true, that this Degree of Skill is not to be known any other way than from Observation; but if the same Observation constantly recur, 'tis strongly to be presumed that a near Estimation of it may be made: However, to make the Calculation more precise, and to avoid causing any needless Scruples to those who love Geometrical Exactness, it will be easy, in the room of the Word Skill, to substitute a Greater or Less Proportion of Chances among the Adventurers, so as each of them may be said to have a certain Number of Chances to win one single Game.

The General Theorem invented by Sir Isaac Newton, for raising a Binomial to any Power given, facilitates infinitely the Method of Combinations, representing in one View the Combination of all the Chances, that can happen in any given Number of Times. 'Tis by the help of that Theorem, joined with some other Methods, that I have been able to find practical Rules for the solving a great Variety of difficult Questions, and to reduce the Difficulty to a single Arithmetical Multiplication, whereof several Instances may be seen in the 21st Page of this Book.

Another Method I have made use of is that of Infinite Series, which in many cases will solve the Problems of Chance more naturally than Combinations. To give the Reader a Notion of this, we may suppose two Men at Play throwing a Die, each in their Turns, and that he be to be reputed the Winner who shall first throw an Ace: It is plain, that the Solution of this Problem cannot so properly be reduced to Combinations, which serve chiefly to determine the proportion of Chances between the Gamesters, without any regard to the Priority of Play. 'Tis convenient therefore to have recourse to some other Method, such as the following. Let us suppose that the first Man, being willing to Compound with his Adversary for the Advantage he is intitled to from his first Throw, should ask him what Consideration he would allow to yield it to him; it may naturally be supposed that the Answer would be one Sixth part of the Stake, there being but Five to One against him, and that this Allowance would be thought a just Equivalent for yielding his Throw: Let us likewise suppose the Second Man to require in his Turn to have one Sixth part of the remaining Stake for the Consideration of his Throw; which being granted, and the first Man's Right returning in course, he may claim again one Sixth part of the Remainder, and so on alternately, till the whole Stake be exhausted:

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But this not being to be done till after an infinite number of Shares be thus taken on both Sides, it belongs to the Method of Infinite Series to assign to each Man what proportion of the Stake he ought to take at first, so as to answer exactly that fictitious Division of the Stake in infinitum; by means of which it will be found, that the Stake ought to be Divided between the contending Parties into two parts, respectively proportional to the two Numbers 6 and 5. By the like Method it would be found that if there were Three or more Adventurers playing on the conditions above described, each Man, according to the Situation he is in with respect to Priority of Play, might take as his Due such part of the Stake, as is expressible by the corresponding Term of the proportion of 6 to 5, continued to so many Terms as there are Gamesters; which in the case of Three Gamesters, for Instance, would be the Numbers 6, 5 and $4\frac{1}{6}$, or their Proportionals 36, 30, and 25.

Another Advantage of the Method of Infinite Series is, that every Term of the Series includes some particular Circumstance wherein the Gamesters may be found, which the other Methods do not; and that a few of its Steps are sufficient to discover the Law of its Process. The only Difficulty which attends this Method, being that of Summing up so many of its Terms as are requisite for the Solution of the Problem proposed: But it will be found by experience, that in the Series resulting from the consideration of most Cases relating to Chance, the Terms of it will either constitute a Geometric Progression, which by the Known Methods is easily Summable; or else some other sort of Progression, whose nature consists in this, that every Term of it has to a determinate number of the preceding Terms, each being taken in order, some constant relation; in which case I have contrived some easie Theorems, not only for finding the Law of that relation, but also for finding the Sums required; as may be seen in several places of this Book, but particularly from page 127 to page 134. I hope the Reader will excuse my not giving the Demonstrations of some few things relating to this Subject, especially of the two Theorems contained in page 134 and 154, and of the Method of Approximation contained in page 149 and 150; whereby the Duration of Play is easily determined with the help of a Table of Natural Sines: Those Demonstrations are omitted purposely to give an occasion to the Reader to exercise his own Ingenuity. In the mean Time, I have deposited them with the Royal Society, in order to be Published when it shall be thought requisite.

A Third Advantage of the Method of Infinite Series is, that the Solutions derived from it have a certain Generality and Elegancy, which scarce any other Method can attain to; those Methods being always perplexed with Various unknown Quantities, and the Solutions obtained by them terminating commonly in particular Cases.

There are other Sorts of Series, which tho' not properly infinite, yet are called Series, from the Regularity of the Terms whereof they are composed; those Terms following one another with a certain uniformity, which is always to be defined. Of this nature is the Theorem given by Sir Isaac Newton, in the Fifth Lemma of the third Book of his Principles, for drawing a Curve through any given number of Points; of which the Demonstration, as well as of other things belonging to the same Subject, may be deduced from the first Proposition of his Methodus Differentialis, printed with some other of his Tracts, by the care of my Intimate Friend, and very skilful Mathematician, Mr. W. Jones. The abovementioned Theorem being very useful in Summing up any number of Terms whose last Differences are equal (Such as are the Numbers called Triangular, Pyramidal, &c. the Squares, the Cubes, or other Powers of Numbers in Arithmetic Progression) I have shewn in many places of this Book how it might be applicable to these Cases. I hope it will not be taken amiss that I have ascribed the Invention of it to its proper Author, tho' 'tis possible some Persons may have found something like it by their own Sagacity.

After having dwelt sometime upon Various Questions depending on the general Principle of Combinations, as laid down in my Introduction, and upon some others depending on the Method of Infinite Series, I proceed to treat of the Method of Combinations properly so called, which I shew to be easily deducible from that more general Principle which hath been before explained: Where it may be observed, that altho' the Cases it is applied to are particular, yet the way of reasoning, and the consequences derived from it, are general; that Method of Arguing about generals by particular Examples, being in my opinion very convenient for easing the Reader's Imagination.

Having explained the common Rules of Combination, and given a Theorem which may be of use for the Solution of some Problems relating to that Subject, I lay down a new Theorem, which is properly a contraction of the former, whereby several Questions of Chance are resolved with wonderful ease, tho' the Solution might seem at first sight to be of insuperable difficulty.

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It is by the Help of that Theorem so contracted, that I have been able to give a compleat Solution of the Problems of Pharaon and Bassete, which was never done till now: I own that some great Mathematicians have before me taken the pains of calculating the Advantage of the Banker, in any circumstance either of Cards remaining in his Hands, or of any number of times that the Card of the Ponte is contained in the Stock: But still the curiosity of the Inquisitive remained unsatisfied; The Chief Question, and by much the most difficult, concerning Pharaon or Bassete, being what it is that the Banker gets per Cent of all the Money adventured at those Games, which now I can certainly answer is very near Three per Cent at Pharaon, and Three fourths per Cent at Bassete, as may be seen in my xxiii Problem, where the precise Advantage is calculated.

In the 24th and 25th Problems, I explain a new sort of Algebra, whereby some Questions relating to Combinations are solved by so easy a Process, that their solution is made in some measure an immediate consequence of the Method of Notation. I will not pretend to say that this new Algebra is absolutely necessary to the Solving of those Questions which I make to depend on it, since it appears by Mr. De Monmort's Book, that both he and Mr. Nicholas Bernouilly have solved, by another Method, many of the cases therein proposed: But I hope I shall not be thought guilty of too much Confidence, if I assure the Reader, that the Method I have followed has a degree of Simplicity, not to say of Generality, which will hardly be attained by any other Steps than by those I have taken.

The 29th Problem, proposed to me, amongst some others, by the Honourable Mr. Francis Robartes, I had solved in my Traité De mensura Sortis; It relates, as well as the 24th and 25th, to the Method of Combinations, and is made to depend on the same Principle; When I began for the first time to attempt its Solution, I had nothing else to guide me but the common Rules of Combinations, such as they had been delivered by Dr. Wallis and others; which when I endeavoured to apply, I was Surprized to find that my calculation swelled by degrees to an Intolerable bulk: For this reason I was forced to turn my Views Another way, and to try whether the solution I was seeking for might not be deduced from some easier considerations; whereupon I happily fell upon the Method I have been mentioning, which as it led me to a very great Simplicity in the Solution, so I look upon it to be an Improvement made to the Method of Combinations.

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The 30th Problem is the reverse of the preceding; It contains a very remarkable Method of Solution, the Artifice of which consists in changing an Arithmetic Progression of Numbers into a Geometric one; this being always to be done when the Numbers are large, and their Intervals small. I freely acknowledge that I have been indebted long ago for this useful Idea, to my much respected Friend, That Excellent Mathematician Doctor Halley, Secretary to the Royal Society, whom I have seen practice the thing on an other occasion: For this and other Instructive Notions readily imparted to me, during an uninterrupted Friendship of five and Twenty years, I return him my very hearty Thanks.

The 32d Problem, having in it a Mixture of the two Methods of Combinations and Infinite Series, may be proposed for a pattern of Solution, in some of the most difficult cases that may occur in the Subject of Chance, and on this occasion I must do that Justice to Mr. Nicholas Bernoulli, the Worthy Professor of Mathematics at Padua, to own he had sent me the Solution of this Problem before mine was Published; which I had no sooner received, but I communicated it to the Royal Society, and represented it as a Performance highly to be commended: Whereupon the Society order'd that his Solution should be Printed; which was accordingly done some time after in the Philosophical Transactions, Numb. 341. where mine was also inserted.

The Problems which follow relate chiefly to the Duration of Play, or to the Method of determining what number of Games may probably be played out by two Adversaries, before a certain number of Stakes agreed on between them be won or lost on either side. This Subject affording a very great Variety of Curious Questions, of which every one has a degree of Difficulty peculiar to it self, I thought it necessary to divide it into several distinct Problems, and to illustrate their Solution with proper Examples.

Tho' these Questions may at first sight seem to have a very great degree of difficulty, yet I have some reason to believe, that the Steps I have taken to come at their Solution, will easily be followed by those who have a competent skill in Algebra, and that the chief Method of proceeding therein will be understood by those who are barely acquainted with the Elements of that Art.

When I first began to attempt the general Solution of the Problem concerning the Duration of Play, there was nothing extant that could give me any light into that Subject; for altho' Mr.
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de Monmort, in the first Edition of his Book, gives the Solution of this Problem, as limited to three Stakes to be won or lost, and farther limited by the Supposition of an Equality of Skill between the Adventurers; yet he having given no Demonstration of his Solution, and the Demonstration when discovered being of very little use towards obtaining the general Solution of the Problem, I was forced to try what my own Enquiry would lead me to, which having been attended with Success, the result of what I found was afterwards published in my Specimen before mentioned.

All the Problems which in my Specimen related to the Duration of Play, have been kept entire in the following Treatise; but the Method of Solution has received some Improvements by the new Discoveries I have made concerning the Nature of those Series which result from the Consideration of the Subject; however, the Principles of that Method having been laid down in my Specimen I had nothing now to do, but to draw the Consequences that were naturally deducible from them.

Mr. de Monmort, and Mr. Nicholas Bernoulli, have each of them separately given the Solution of my xxxixth Problem, in a Method differing from mine, as may be seen in Mr. de Monmort's second Edition of his Book. Their Solutions, which in the main agree together, and vary little more than in the form of Expression, are extremely beautiful; for which reason I thought the Reader would be well pleased to see their Method explained by me, in such a manner as might be apprehended by those who are not so well versed in the nature of Symbols: In which matter I have taken some Pains, thereby to testify to the World the just Value I have for their Performance.

The 43d Problem having been proposed to me by Mr. Thomas Woodcock, a Gentleman whom I infinitely respect, I attempted its Solution with a very great desire of obtaining it; and having had the good Fortune to succeed in it, I returned him the Solution a few Days after he was pleased to propose it. This Problem is in my Opinion one of the most curious that can be propos'd on this Subject; its Solution containing the Method of determining, not only that Advantage which results from a Superiority of Chance, in a Play confined to a certain number of Stakes to be won or lost by either Party, but also that which may result from an unequality of Stakes; and even compares those two Advantages together, when the Odds of Chance being on one side, the Odds of Money are on the other.

Before I make an end of this Discourse, I think my self obliged to take Notice, that some Tears after my Specimen was printed, there came out a Tract upon the Subject of Chance, being a Posthumous Work of Mr. James Bernoulli, wherein the Author has shewn a great deal of Skill and Judgment, and perfectly answered the Character and great Reputation he hath so justly obtained. I wish I were capable of carrying on a Project he had begun, of applying the Doctrine of Chances to Oeconomical and Political Uses, to which I have been invited, together with Mr de Monmort, by Mr. Nicholas Bernoulli : I heartily thank that Gentleman for the good Opinion he has of me ; but I willingly resign my share of that Task into better Hands, wishing that either he himself would prosecute that Design, he having formerly published some successful Essays of that Kind, or that his Uncle, Mr. John Bernoulli, Brother to the Deceased, could be prevailed upon to bestow some of his Thoughts upon it ; he being known to be perfectly well qualified in all Respects for such an Undertaking.

Due Care having been taken to avoid the *Errata* of the Press, we hope there are no other than these two,

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Pag. 35. Lin. 35. for $n - 3$ read $n - 1$.

Pag. 36. Lin. 2. for $n - 3$ read $n - 2$.



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The DOCTRINE OF CHANCES.

INTRODUCTION.



HE Probability of an Event is greater, or less, according to the number of Chances by which it may Happen, compar'd with the number of all the Chances, by which it may either Happen or Fail.

Thus, If an Event has 3 Chances to Happen, and 2 to Fail; the Probability of its Happening may be estimated to be $\frac{3}{5}$, and the Probability of its Failing $\frac{2}{5}$.

Therefore, if the Probability of Happening and Failing are added together, the Sum will always be equal to Unity.

B

If

If the Probabilities of Happening and Failing are unequal, there is what is commonly call'd Odds for, or against, the Happening or Failing; which Odds are proportional to the number of Chances for Happening or Failing.

The Expectation of obtaining any Thing, is estimated by the Value of that Thing multiplied by the Probability of obtaining it.

Thus, Supposing that *A* and *B* Play together; that *A* has deposited 5 *l*, and *B* 3 *l*; that the number of Chances which *A* has to win is 4, and that the number of Chances which *B* has to win is 2: Since the whole Sum deposited is 8, and that the Probability which *A* has of getting it, is $\frac{4}{6}$; it follows, that the Expectation of *A* upon the whole Sum deposited will be $\frac{8}{1} \times \frac{4}{6} = 5 \frac{1}{3}$; and for the same reason, the Expectation of *B* will be $\frac{8}{1} \times \frac{2}{6} = 2 \frac{2}{3}$.

The Risk of losing any Thing, is estimated by the Value of that Thing multiplied by the Probability of losing it.

If from the respective Expectations, which the Gamesters have upon the whole Sum deposited; the particular Sums they deposit, that is their own Stakes, be subtracted, there will remain the Gain, if the difference is positive, or the Loss, if the difference is negative.

Thus, If from $5 \frac{1}{3}$ the Expectation of *A*, 5 which is his own Stake be subtracted, there will remain $\frac{1}{3}$ for his Gain; likewise if from $2 \frac{2}{3}$ the Expectation of *B*, 3 which is his own Stake be subtracted, there will remain $-\frac{1}{3}$ for his Gain, or $\frac{1}{3}$ for his Loss.

Again, If from the respective Expectations, which either Gamester has upon the Sum deposited by his Adversary, the Risk of losing what he himself deposits, be subtracted, there will likewise remain his Gain or Loss.

Thus, In the preceding Case, the Stake of *B* being 3, and the Probability which *A* has of winning it being $\frac{4}{6}$, the Expectation of *A* upon that Stake is $\frac{3}{1} \times \frac{4}{6} = \frac{12}{6} = 2$. Moreover the Stake of *A* being 5, and the Probability of losing it being $\frac{2}{6}$, the Risk which *A* runs of losing his own Stake is $\frac{5}{1} \times \frac{2}{6} = \frac{10}{6} = 1 \frac{2}{3}$. Therefore, if from the Ex-

Expectation 2, the Risk $1 \frac{2}{3}$, be subtracted, there will remain $\frac{1}{3}$, as before, for the Gain of *A*; and by the same way of arguing, the Loss of *B* will be found to be $\frac{1}{3}$.

N. B. Tho' the Gain of one is the Loss of the other, yet it will be convenient to look for them severally, that one Operation may be a Proof of the other.

If there is a certain number of Chances by which the possession of a Sum can be secur'd; and also a certain number of Chances by which it may be lost; that Sum may be Insured for that part of it, which shall be to the whole, as the number of Chances there is to lose it, to the number of all the Chances.

Thus, If there are 19 Chances to secure the possession of 1000*l*, and 1 Chance to lose it, the Insurance Money may be found by this Proportion.

As 20 is to 1, so is 1000 to 50; therefore 50 is the Sum that ought to be given, in this Case, to Insure 1000.

If two Events have no dependence on each other, so that *p* be the number of Chances by which the first may Happen, and *q* the number of Chances by which it may Fail; and likewise that *r* be the number of Chances by which the second may Happen, and *s* the number of Chances by which it may Fail: Multiply *p* + *q* by *r* + *s*, and the Product *pr* + *qr* + *ps* + *qs* will contain all the Chances, by which the Happening, or Failing of the Events may be varied amongst one another.

Therefore, If *A* and *B* Play together, on condition that if both Events Happen, *A* shall win, and *B* lose; the Odds that *A* shall be the winner, are as *pr* to *qr* + *ps* + *qs*; for the only Term in which both *p* and *r* occur is *pr*; therefore the Probability of *A*'s winning is $\frac{pr}{pr+qr+ps+qs}$.

But if *A* holds that either one or the other will Happen; the Odds of *A*'s winning are as *pr* + *qr* + *ps* to *qs*; for some of the Chances that are favourable to *A*, occur in every one of the Terms *pr*, *qr*, *ps*.

Again, If *A* holds that the first will Happen, and the second Fail; the Odds are as *ps* to *pr* + *qr* + *qs*.

From

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From what has been said, it follows, that if a Fraction expresses the Probability of an Event, and another Fraction the Probability of another Event, and those two Events are independent; the Probability that both those Events will Happen, will be the Product of those two Fractions.

Thus, Suppose I have two Wagers depending, in the first of which I have 3 to 2 the best of the Lay, and in the second 7 to 4, what is the Probability I win both Wagers?

The Probability of winning the first is $\frac{3}{5}$, that is the number of Chances I have to win, divided by the number of all the Chances; the Probability of winning the second is $\frac{7}{11}$: Therefore multiplying these two Fractions together, the Product will be $\frac{21}{55}$, which is the Probability of winning both Wagers. Now this Fraction being subtracted from 1, the remainder is $\frac{34}{55}$, which is the Probability I do not win both Wagers: Therefore the Odds against me are 34 to 21.

2° If I would know what the Probability is of winning the first, and losing the second, I argue thus; The Probability of winning the first is $\frac{3}{5}$, the Probability of losing the second is $\frac{4}{11}$: Therefore multiplying $\frac{3}{5}$ by $\frac{4}{11}$, the Product $\frac{12}{55}$ will be the Probability of my winning the first, and losing the second; which being subtracted from 1, there will remain $\frac{43}{55}$, which is the Probability I do not win the first, and at the same time lose the second.

3° If I would know what the Probability is of winning the second, and at the same time losing the first; I say thus, the Probability of winning the second is $\frac{7}{11}$, the Probability of losing the first is $\frac{2}{5}$. Therefore multiplying these two Fractions together, the Product $\frac{14}{55}$ is the Probability I win the second, and also lose the first.

4° If I would know what the Probability is of losing both Wagers; I say, the Probability of losing the first is $\frac{2}{5}$, and the Probability of losing the second $\frac{4}{11}$; therefore the Probability of losing them both is $\frac{8}{55}$, which being subtracted from 1, there remains $\frac{47}{55}$; therefore the Odds against losing both Wagers is 47 to 8.

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This way of reasoning is plain, and is of very great extent, being applicable to the Happening or Failing of as many Events as may fall under consideration. Thus, if I would know what the Probability is of missing an Ace 4 times together with a common Die, I consider the missing of the Ace 4 times, as the Failing of 4 different Events; now the Probability of missing the first is $\frac{5}{6}$, the Probability of missing the second is also $\frac{5}{6}$, the third $\frac{5}{6}$, the fourth $\frac{5}{6}$; therefore the Probability of missing the Ace 4 times together is $\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{625}{1296}$; which being subtracted from 1, there will remain $\frac{671}{1296}$ for the Probability of throwing it once or oftner in 4 times; therefore the Odds of throwing an Ace in 4 times is 671 to 625.

But if the flinging of an Ace was undertaken in 3 times, 'tis plain that the Probability of missing it 3 times would be $\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{125}{216}$, which being subtracted from 1, there will remain $\frac{91}{216}$ for the Probability of throwing it once or oftner in 3 times; therefore the Odds against throwing it in 3 times are 125 to 91.

Again, suppose we wou'd know the Probability of throwing an Ace once in 4 throws and no more: Since the Probability of throwing it the first time is $\frac{1}{6}$, and the Probability of missing it the other three times is $\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}$, it follows that the Probability of throwing it the first time, and missing it afterwards three times successively, is $\frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{125}{1296}$; but because it is possible to hit it in every throw as well as the first, it follows, that the Probability of throwing it once in 4 throws, and missing the other three times, is $\frac{4 \times 125}{1296} = \frac{500}{1296}$; which being subtracted from 1, there will remain $\frac{796}{1296}$ for the Probability of not throwing it once and no more in 4 times; therefore if one undertakes to throw an Ace once and no more in 4 times, he has 500 to 796 the worst of the Lay, or 5 to 8 very near.

Suppose two Events are such, that one of them has twice as many Chances to come up as the other; what is the Probability that the Event which has the greater number of Chances to come up, does not Happen twice before the other Happens once; which is the Case of flinging Seven with two Dice, be-

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fore four once ; since the number of Chances are as 2 to 1, the Probability of the first Happening before the second is $\frac{2}{3}$, but the Probability of its Happening twice before it, is but $\frac{2}{3} \times \frac{2}{3}$ or $\frac{4}{9}$; therefore 'tis 5 to 4, Seven does not come up twice, before Four once.

But if it was demanded what must be the proportion of the Facilities of the coming up of two Events, to make that which has the most Chances, to come up twice, before the other comes up once ; the answer is 12 to 5 very near, (and this proportion may be determined yet with greater exactness) for if the proportion of the Chances is 12 to 5, it follows, that the Probability of throwing the first before the second is $\frac{12}{17}$, and the Probability of throwing it twice, is $\frac{12}{17} \times \frac{12}{17}$ or $\frac{144}{289}$; therefore the Probability of not doing it is $\frac{145}{289}$; therefore the Odds against it are as 145 to 144, which comes very near a proportion of Equality.

What we have said hitherto concerning two or more Events, relates only to those which have no dependency on each other ; as for those that have a dependency, the manner of arguing about them will be a little alter'd : But to know in what the nature of this dependency consists, I shall propose the two following easy Problems.

Suppose there is a heap of 13 Cards of one colour, and another heap of 13 Cards of another colour ; what is the Probability, that taking one Card at a venture out of each heap, I shall take out the two Aces ?

The Probability of taking the Ace out of the first heap is $\frac{1}{13}$, the Probability of taking the Ace out of the second is also $\frac{1}{13}$; therefore the Probability of taking out both Aces is $\frac{1}{13} \times \frac{1}{13} = \frac{1}{169}$, which being subtracted from 1, there will remain $\frac{168}{169}$, therefore the Odds against me are 168 to 1.

But suppose that out of one single heap of 13 Cards of one colour, I should undertake to take out, first the Ace, secondly the Two ; tho' the Probability of taking out the Ace be $\frac{1}{13}$, and the Probability of taking out the Two be likewise $\frac{1}{13}$, yet the Ace being supposed as taken out already, there will remain only 12 Cards in the heap, which will make the Probability of

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of taking out the Two to be $\frac{1}{12}$, therefore the Probability of taking out the Ace, and then the Two, will be $\frac{1}{13} \times \frac{1}{12}$. And upon this way of reasoning may the whole Doctrine of Combinations be grounded, as will be shewn in its place.

It is plain that in this last Question, the two Events proposed have on each other a dependency of Order, which dependency consists in this, that one of the Events being supposed as having Happened, the Probability of the other's Happening is thereby alter'd; whereas in the first Question, the taking of the Ace out of the first heap does not alter the Probability of taking the Ace out of the second; therefore the Independency of Events consists in this, that the Happening of one does not alter the degree of Probability of the other's Happening.

We have seen already how to determine the Probability of the Happening of as many Events as may be assigned, and the Failing of as many others as may be assigned likewise, when those Events are independent: We have seen also how to determine the Happening of two Events, or as many as may be assigned when they are Dependent.

But how to determine in the case of Events dependent, the Happening of as many as may be assigned, and at the same time the Failing of as many as may likewise be assigned, is a disquisition of a higher nature, and will be shewn afterwards.

If the Events in question are n in number, and are such as have the same number a of Chances by which they may Happen, and likewise the same number b of Chances by which they may Fail, raise $a + b$ to the Power n .

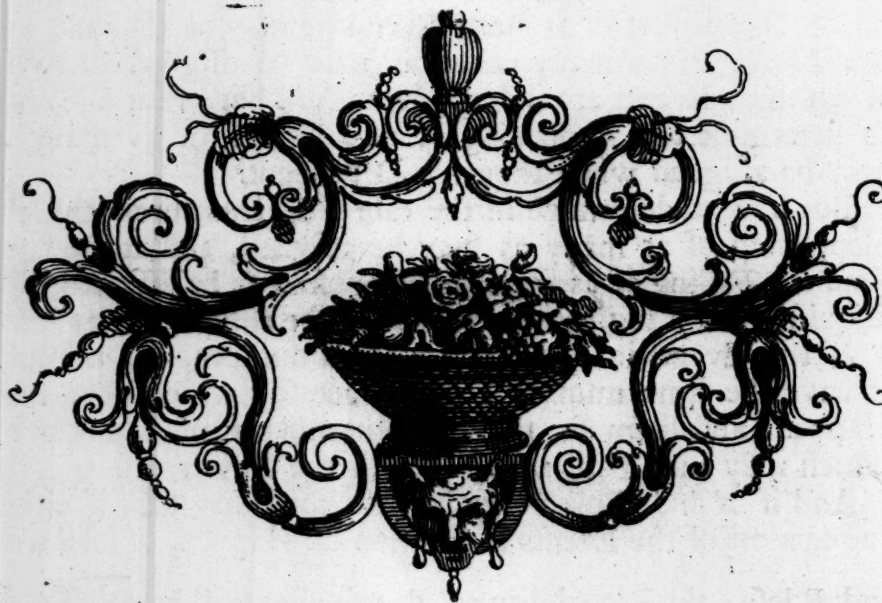
And if A and B play together, on condition that if either one or more of the Events in question do Happen, A shall win, and B lose; the Probability of A 's winning will be $\frac{a + b^n - b^n}{a + b^n}$,

and that of B 's winning will be $\frac{b^n}{a + b^n}$; for when $a + b$ is actually raised to the Power n , the only Term in which a does not occur is the last b^n ; therefore all the Terms but the last are favourable to A .

Thus, if $n = 3$; raising $a + b$ to the Cube $a^3 + 3aab + 3abb + b^3$. all the Terms but b^3 will be favourable to A ; and therefore
the

the Probability of *A*'s winning will be $\frac{a^3 + 3aab + 3abb}{a+b^3}$ or $\frac{a+b^3-b^3}{a+b^3}$; and the Probability of *B*'s winning will be $\frac{b^3}{a+b^3}$.

But if *A* and *B* play, on condition that if either two or more of the Events in question do Happen, *A* shall win; but in case one only Happens or none, *B* shall win; the Probability of *A*'s winning will be $\frac{a+b^n-nab^{n-1}-b^n}{a+b^n}$; for the only two Terms in which *aa* does not occur are the two last, viz, nab^{n-1} and b^n . And so of the rest.



*The Solution of several sorts of Problems,
deduced from the Rules laid down in the
Introduction.*

PROBLEM I.



Suppose A to hold a single Die, and to lay with B, that in 8 throws he shall fling Two Aces or more: What is his Probability of winning, or what are the Odds for or against him?

SOLUTION.

Because there is one single Chance for A, and five against him, let a be made $= 1$, and $b = 5$; again because the number of throws is 8, let n be made $= 8$, and the Probability of A's winning will be $\frac{a+b^n - b^n - nab^{n-1}}{a+b^n} = \frac{663991}{1679616}$

Therefore the Probability of his losing will be $\frac{1015625}{1679616}$, and the Odds against him will be as 1015625 to 663991, or as 3 to 2, very near.

PROBLEM II.

TWO Men A and B playing a Set together, in each Game of the Set the number of Chances which A has to win is 3, and the number of Chances which B has to win is 2: Now after some Games are over, A wants 4 Games of being up, and B 6: It is required in this circumstance to determine the Probabilities which either has of winning the Set.

SOLUTION.

Because A wants 4 Games of being up, and B 6; it follows, that the Set will be ended in 9 Games at most, which is the sum of the Games wanting between them;

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them; therefore let $a + b$ be raised to the 9th Power, viz. $a^9 + 9a^8b + 36a^7bb + 84a^6b^3 + 126a^5b^4 + 126a^4b^5 + 84a^3b^6 + 36aab^7 + 9ab^8 + b^9$; take for A all the Terms in which a has 4 or more dimensions, and for B all the Terms in which b has 6 or more dimensions; and the Proportion of the Odds will be, as $a^9 + 9a^8b + 36a^7bb + 84a^6b^3 + 126a^5b^4 + 126a^4b^5$, to $84a^3b^6 + 36aab^7 + 9ab^8 + b^9$. Let now a be expounded by 3, and b by 2; and the Odds that A wins the Set will be found as 1759077 to 194048, or very near as 9 to 1.

And generally, supposing that p and q are the number of Games respectively wanting; raise $a + b$ to the Power $p + q - 1$, then take for A , a number of Terms equal to q , and for B , a number of Terms equal to p .

REMARKS.

1. In this Problem, if instead of supposing that the Chances which the Gamesters have each time to get a Game are in the proportion of a to b , we suppose the Skill of the Gamesters to be in that proportion, the Solution of the Problem will be the same: We may compare the Skill of the Gamesters to the number of Chances they have to win. Whether the number of Chances which A and B have of getting a Game, are in a certain Proportion, or whether their Skill be in that proportion, is the same thing.

2. The preceding Problem might be solved without Algebra, by the bare help of the Arithmetical Principles which we have laid down in the Introduction, but the method will be longer: Yet for the sake of those who are not acquainted with Algebraical computation, I shall set down the Method of proceeding in like cases.

In order to which, it is necessary to know, that when a Question seems somewhat difficult, it will be useful to solve at first a Question of the like nature, that has a greater degree of simplicity than the case proposed in the Question given; the Solution of which case being obtained, it will be a step to ascend to a case a little more compounded, till at last the case proposed may be attained to.

Therefore, to begin with the simplest case, we may suppose that A wants 1 Game of being up, and B 2; and that the

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the number of Chances to win a Game are equal ; in which case the Odds that *A* will be up before *B*, may be determined as follows.

Since *B* wants 2 Games of being up and *A* 1, 'tis plain that *B* must beat *A* twice together to win; but the Probability of his beating him once is $\frac{1}{2}$, therefore the Probability of his beating him twice together is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$; subtract $\frac{1}{4}$ from 1, there remains $\frac{3}{4}$, which is the Probability which *A* has of winning once before *B* twice, therefore the Odds are as 3 to 1.

By the same way of arguing 'twill be found, that if *A* wants 1, and *B* 3, the Odds will be as 7 to 1, and the Probability of winning, $\frac{7}{8}$ and $\frac{1}{8}$ respectively. If *A* wants 1, and *B* 4, the Odds will be as 15 to 1, &c.

Again, suppose *A* wants 2, and *B* 3, what are the Odds that *A* is up before *B*?

Let the whole Stake deposited between *A* and *B* be 1; now consider that if *B* wins the first Game, *B* and *A* will have an equality of Chances, in which case the Expectation of *B* will be $\frac{1}{2}$; but the Probability of his winning the first Game is $\frac{1}{2}$, therefore the Expectation of *B* upon the Stake, arising from the Probability of beating *A* the first time, will be $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

But if *B* loses the first Game, then he will want 3 of being up, and *A* but 1; in which case the Expectation of *B* will be $\frac{1}{8}$, but the Probability of that circumstance is $\frac{1}{2}$, therefore the Expectation of *B* arising from the Probability of his losing the first time is $\frac{1}{2} \times \frac{1}{8} = \frac{1}{16}$.

Therefore the Expectation of *B* upon the Stake 1, will be $\frac{1}{4} + \frac{1}{16} = \frac{5}{16}$, which being subtracted from 1, there remains $\frac{11}{16}$ for the Expectation of *A*; therefore the Odds are as 11 to 5.

And thus proceeding gradually, it will be easy to compose the following Table.

A TABLE.

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A TABLE of the ODDS for any number of Games wanting, from 1 to 6.

Games wanting.	Probabilities of Winning.	Games wanting.	Probabilities of Winning.	Games wanting.	Probabilities of Winning.	Games wanting.	Probabilities of Winning.	Games wanting.	Probabilities of Winning.
I	$\frac{3}{4}$	2	$\frac{1}{4}$	I	$\frac{7}{8}$	3	$\frac{1}{8}$	I	$\frac{15}{16}$
2	$\frac{11}{16}$	3	$\frac{5}{16}$	2	$\frac{26}{32}$	4	$\frac{6}{32}$	2	$\frac{57}{64}$
3	$\frac{42}{64}$	4	$\frac{22}{64}$	3	$\frac{99}{128}$	5	$\frac{29}{128}$	3	$\frac{219}{256}$
4	$\frac{163}{256}$	5	$\frac{93}{256}$	4	$\frac{382}{512}$	6	$\frac{130}{512}$	4	$\frac{37}{256}$
5	$\frac{638}{1024}$	6	$\frac{386}{1024}$					5	$\frac{31}{32}$
								6	$\frac{1}{64}$

And by the same way of proceeding, it would be easy to compose other Tables, for expressing the Probabilities which *A* and *B* have of winning the Set, when each wants a given number of Games of being up, and when the proportion of the Chances

Chances by which each of them may get a Game is as 2 to 1, or varies at pleasure; but the Algebraic method explained in this Problem answering all that variety, 'tis needless to insist upon it.

PROBLEM III.

IF A and B play with single Bowls, and such be the Skill of A that he knows by Experience he can give B 2 Games out of 3: What is the proportion of their Skill, or what are the Odds that A may get any one Game assigned?

SOLUTION.

LET the proportion of the Odds be as z to 1: Now since A can give B 2 Games out of 3, therefore A can, upon an equality of Play, undertake to win 3 Games together: Let therefore $z + 1$ be raised to the Cube, viz. $z^3 + 3zz + 3z + 1$; therefore the Probabilities of winning will be, as z^3 to $3zz + 3z + 1$; but these Probabilities are equal, by supposition; therefore $z^3 = 3zz + 3z + 1$, or $2z^3 = z^3 + 3zz + 3z + 1$. and extracting the Cube Root on both sides, $z\sqrt[3]{2} = z + 1$; therefore $z = \frac{1}{\sqrt[3]{2}-1}$, and consequently the Odds that A may get any one Game assigned are as $\frac{1}{\sqrt[3]{2}-1}$ to 1, or as 1 to $\sqrt[3]{2}-1$, that is in this case as 50 to 13 very near.

PROBLEM IV.

IF A can without advantage or disadvantage give B 1 Game out of 3; what are the Odds that A shall take any one Game assigned? Or what is the proportion of the Chances they have to win any one Game assigned? Or what is the proportion of their Skill?

SOLUTION.

LET the proportion be as z to 1; now since A can give B 1 Game out of 3; therefore A can, upon an equality of Play, undertake to get 3 Games before B gets 2; let there-
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fore $z+1$ be raised to the 4th Power, whose Index 4 is the Sum of the Games wanting between them less by 1; this Power will be $z^4 + 4z^3 + 6z^2 + 4z + 1$; therefore the Probabilities of winning the Set will be as $z^4 + 4z^3$ to $6z^2 + 4z + 1$: But these Probabilities are equal by Hypothesis, since A and B are supposed to play without advantage or disadvantage; therefore $z^4 + 4z^3 = 6z^2 + 4z + 1$, which Equation being solved, z will be found to be 1.6 very near; wherefore the proportion of the Odds will be as 1.6 to 1, or as 8 to 5.

PROBLEM V.

TO find in how many Trials an Event will Probably Happen, or how many Trials will be requisite to make it indifferent to lay on its Happening or Failing; supposing that a is the number of Chances for its Happening in any one Trial, and b the number of Chances for its Failing.

SOLUTION.

LET x be the number of Trials; therefore by what has been already demonstrated in the Introduction $\frac{a}{a+b}^x - \frac{b}{a+b}^x = b^x$, or $\frac{a}{a+b}^x = 2b^x$; therefore $x = \frac{\text{Log } 2}{\text{Log } \frac{a}{a+b} - \text{Log } b}$.

Moreover, let us reassume the Equation $\frac{a}{a+b}^x = 2b^x$ and making $a, b :: 1, q$, the Equation will be changed into this $\frac{1}{1+\frac{1}{q}}^x = 2$: let therefore $1 + \frac{1}{q}$ be raised actually to the Power x by Sir Isaac Newton's Theorem, and the Equation will be $1 + \frac{x}{q} + \frac{x \times x - 1}{1 \times 2 \times q \times q} + \frac{x \times x - 1 \times x - 2}{1 \times 2 \times 3 \times q^3} \&c. = 2$. In this Equation, if $q = 1$, then will x be likewise $= 1$; if q be infinite, then will x also be infinite. Suppose q infinite, then the Equation will be reduced to $1 + \frac{x}{q} + \frac{x^2}{2qq} + \frac{x^3}{6qqq} \&c. = 2$: But the first part of this Equation is the number whose Hyperbolic Logarithm is $\frac{x}{q}$, therefore $\frac{x}{q} = \text{Log } 2$: But the Hyperbolic Logarithm of 2 is 0.693 or nearly 0.7; Wherefore $\frac{x}{q} = 0.7$, and $x = 0.7q$ very near.

Thus we have assigned the very narrow limits within which the Ratio of x to q is comprehended; for it begins with

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with Unity, and terminates at last in the Ratio of 10 to 7, very near.

But x soon Converges to the limit $0.7q$, so that this proportion may be assumed in all cases, let the Value of q be what it will.

Some uses of this Proposition will appear by the following Examples.

EXAMPLE I.

LET it be proposed to find in how many Throws one may undertake, with an equality of Chance, to fling two Aces with two Dice.

The number of Chances upon two Dice is 36, out of which there is but 1 Chance for two Aces; therefore the number of Chances against it is ~~35~~³⁶; Multiply 35 by 0.7, and the product 24.5 will shew that the number of Throws requisite to that effect will be between 24 and 25.

EXAMPLE II.

TO find in how many Throws of three Dice, one may undertake to fling three Aces.

The number of all the Chances upon 3 Dice is 216, out of which there is but 1 Chance for 3 Aces, and 215 against it. Therefore let 215 be multiplied by 0.7, and the product 150.5 will shew that the number of Chances requisite to that effect will be 150, or very near it.

EXAMPLE III.

IN a Lottery whereof the number of Blanks is to the number of Prizes as 39 to 1, (such as was the Lottery of 1710;) To find how many Tickets one must take, to make it an equal Chance for one or more Prizes.

Multiply 39 by 0.7, and the product 27.3 will show that the number of Tickets requisite to that effect will be 27, or 28 at most.

Likewise, in a Lottery whereof the number of Blanks is to the number of Prizes; as 5 to 1, multiply 5 by 0.7 and the pro-

5.07
3.50

product 3.5 will show, that there is more than an equality of Chance in 4 Tickets for one or more Prizes, but something less than an equality in 3.

R E M A R K.

In a Lottery whereof the Blanks are to the Prizes as 39 to 1, if the number of Tickets in all was but 40, this proportion would be altered, for 20 Tickets would be a sufficient number for the Expectation of the single Prize; it being evident that the Prize may be as well among the Tickets which are taken as among those that are left behind.

Again, if the number of Tickets was 80, still preserving the proportion of 39 Blanks to 1 Prize, and consequently supposing 78 Blanks to 2 Prizes, this proportion would still be altered: For by the Doctrine of Combinations, whereof we are to treat afterwards, it will appear that the Probability of taking one Prize or both in 20 Tickets would be but $\frac{139}{316}$, and the Probability of taking none would be $\frac{177}{316}$; Wherefore the Odds against taking any Prize would be as 177 to 139, or very near as 9 to 7.

And by the same Doctrine of Combinations it will be found that 23 Tickets would not be quite sufficient for the Expectation of a Prize in this Lottery; but that 24 would rather be too many; so that one might with advantage lay an even Wager of taking a Prize in 24 Tickets.

If the proportion of 39 to 1 be oftner repeated, the number of Tickets requisite for a Prize will still increase with that repetition: Yet let the proportion of 39 to 1 be repeated never so many times, nay an infinite number of times, the number of Tickets requisite for a Prize will never exceed $\frac{7}{10}$ of 39, that is about 27 or 28.

Therefore if the proportion of the Blanks to the Prizes be often repeated, as it usually is in Lotteries; the number of Tickets requisite for one Prize or more, will always be found by taking $\frac{7}{10}$ of the proportion of the Blanks to the Prizes.

L E M.

LEMMA.

TO find how many Chances there are upon any number of Dice, each of them of the same given number of Faces, to throw any given number of Points.

SOLUTION.

LET $p+1$ be the number of Points given, n the number of Dice, f the number of Faces in each Die: Make $p-f=q$; $q-f=r$; $r-f=s$; $s-f=t$ &c. and the number of Chances will be

$$\begin{aligned} &+ \frac{p}{1} \times \frac{p-1}{2} \times \frac{p-2}{3} \text{ \&c.} \\ &- \frac{q}{1} \times \frac{q-1}{2} \times \frac{q-2}{3} \text{ \&c.} \times \frac{n}{1} \\ &+ \frac{r}{1} \times \frac{r-1}{2} \times \frac{r-2}{3} \text{ \&c.} \times \frac{n}{1} \times \frac{n-1}{2} \\ &- \frac{s}{1} \times \frac{s-1}{2} \times \frac{s-2}{3} \text{ \&c.} \times \frac{n}{4} \times \frac{n-1}{2} \times \frac{n-3}{3} \\ &+ \text{\&c.} \end{aligned}$$

which Series ought to be continued till some of the Factors in each Product become either = 0, or Negative.

N. B. So many Factors are to be taken in each of the Products $\frac{p}{1} \times \frac{p-1}{2} \times \frac{p-2}{3} \text{ \&c.}$ $\frac{q}{1} \times \frac{q-1}{2} \times \frac{q-2}{3} \text{ \&c.}$ as there are Units in $n-1$.

Thus, for Example, let it be required to find how many Chances there are for throwing Sixteen Points with Four Dice.

$$\begin{aligned} &+ \frac{15}{1} \times \frac{14}{2} \times \frac{13}{3} &= + 455 \\ &- \frac{9}{1} \times \frac{8}{2} \times \frac{7}{3} \times \frac{4}{1} &= - 336 \\ &+ \frac{3}{1} \times \frac{2}{2} \times \frac{1}{3} \times \frac{4}{1} \times \frac{3}{2} &= + 6 \end{aligned}$$

But $455 - 336 + 6 = 125$; therefore One Hundred and Twenty Five is the number of Chances required.

F

Again,

Again, let it be required to find the number of Chances for throwing seven and Twenty Points with Six Dice.

$$\begin{aligned}
 &+ \frac{26}{1} \times \frac{25}{2} \times \frac{24}{3} \times \frac{23}{4} \times \frac{22}{5} &= + 65780 \\
 &- \frac{19}{1} \times \frac{18}{2} \times \frac{17}{3} \times \frac{16}{4} \times \frac{15}{5} \times \frac{6}{1} &= - 93204 \\
 &+ \frac{13}{1} \times \frac{12}{2} \times \frac{11}{3} \times \frac{10}{4} \times \frac{9}{5} \times \frac{6}{1} \times \frac{5}{2} &= + 30030 \\
 &- \frac{8}{1} \times \frac{7}{2} \times \frac{6}{3} \times \frac{5}{4} \times \frac{4}{3} \times \frac{6}{1} \times \frac{5}{2} \times \frac{4}{3} &= - 1120
 \end{aligned}$$

Therefore $65780 - 93204 + 30030 - 1120 = 1666$ is the number required.

Let it be required to find the number of Chances for throwing Fifteen Points with Six Dice.

$$\begin{aligned}
 &+ \frac{14}{1} \times \frac{13}{2} \times \frac{12}{3} \times \frac{11}{4} \times \frac{10}{5} &= + 2002 \\
 &- \frac{8}{1} \times \frac{7}{2} \times \frac{6}{3} \times \frac{5}{4} \times \frac{4}{5} \times \frac{6}{1} &= - 336
 \end{aligned}$$

But $2002 - 336 = 1666$, which is the number required.

Corol. All the Points equally distant from the extreams, that is, from the least and greatest number of Points that are upon the Dice, have the same number of Chances by which they may be produced; wherefore if the number of Points given be nearer to the greater Extream than to the less, let the number of Points given be subtracted from the sum of the Extreams, and work with the remainder, and the Operation will be shorten'd.

Thus, if it be required to find the number of Chances for throwing 27 Points with 6 Dice: Let 27 be subtracted from 42 the sum of the Extreams 6 and 36, and the Remainder being 15, it may be concluded that the number of Chances for throwing 27 Points is the same as for throwing 15.

Let it now be required to find in how many throws of 6 Dice one may undertake to throw 15 Points.

The number of Chances for throwing 15 Points being 1666; and the number of Chances for Failing being 44990; divide 44990 by 1666, the Quotient will be 27; Multiply 27 by

by 0.7, and the Product 18.9 will shew that the number of throws requisite to that effect is very near 19.

PROBLEM VI.

TO find how many Trials are necessary to make it Probable, that an Event will Happen twice, supposing that a is the number of Chances for its Happening at any one Trial, and b the number of Chances for its Failing.

SOLUTION.

LET x be the number of Trials: Therefore by what has been already Demonstrated, it will appear that $a + b^x = 2b^x + 2axb^{x-1}$; or making $a, b :: 1, q$; $1 + \frac{1}{q}^x = 2 + \frac{2x}{q}$. Now if q be supposed $= 1$, x will be found $= 3$; and if q be supposed infinite, and also $\frac{x}{q} = z$, we shall have $z = \text{Log: } 2 + \text{Log: } 1 + z$; in which Equation the value of z will be found $= 1.678$ very nearly. Therefore the value of x will always be between the Limits $3q$ and $1.678q$. But x will soon converge to the last of these Limits; therefore if x be not very small, it may in all cases be supposed $= 1.678q$. Yet if there be any Suspicion that the Value of x thus taken is too little, substitute this Value in the Original Equation $1 + \frac{1}{q}^x = 2 + \frac{2x}{q}$, and note that Errour. If it be worth taking notice of, then increase a little the value of x , and substitute again this new value in the room of x in the aforesaid Equation; and noting the new Errour, the value of x may be sufficiently corrected by applying the Rule which the Arithmeticians call double False Position.

EXAMPLE I.

TO find in how many throws of Three Dice one may undertake to throw Three Aces twice.

The number of all the Chances upon Three Dice being 216, out of which there is but one Chance for Three Aces, and 215 against it; Multiply 215 by 1.678, and the Product 360.7 will shew that the number of throws requisite to that effect will be 360 or very near it.

EX-

EXAMPLE II.

TO find in how many throws of Six Dice one may undertake to throw Fifteen Points twice.

The number of Chances for throwing Fifteen Points is 1666, the number of Chances for Missing 44990; let 44990 be divided by 1666, the Quotient will be 27 very near: Wherefore the proportion of Chances for Throwing and Missing Fifteen Points are as 1 to 27 respectively; Multiply 27 by 1.678, and the Product 45.3 will shew that the number of throws requisite to that effect will be 45 nearly.

EXAMPLE III.

IN a Lottery whereof the number of Blanks is to the number of Prizes as 39 to 1: To find how many Tickets must be taken, to make it as Probable that two or more Benefits will be taken as not.

Multiply 39 by 1.678, and the Product 65.4 will shew that no less than 65 Tickets will be requisite to that effect; tho' one might undertake upon an Equality of Chance to have one at least in 28.

PROBLEM VII.

TO find how many Trials are necessary to make it Probable that an Event will Happen Three, Four, Five, &c. times; supposing that a is the number of Chances for its Happening in any one Trial, and b the number of Chances for its Failing.

SOLUTION.

LET x be the number of Trials requisite, then supposing, as before $a, b :: 1, q$, we shall have the Equation $1 + \frac{1}{q} x = 2 \times 1 + \frac{x}{q} + \frac{x}{1} \times \frac{x-1}{2qq}$, in the case of the triple Event; or $1 + \frac{1}{q} x = 2 \times 1 + \frac{x}{q} + \frac{x}{1} \times \frac{x-1}{2qq} + \frac{x}{1} \times \frac{x-1}{2} \times \frac{x-2}{3qq}$, in the case of the quadruple Event: And the Law of the continuation of these Equations is manifest. Now in the first Equation if q be supposed = 1, then will x be = 5. If q be sup-

supposed infinite or pretty large in respect to Unity; then the aforesaid Equation, making $\frac{x}{q} = z$, will be changed into this, $z = \text{Log. } 2 + \text{Log. } \frac{1+z+\frac{1}{2}zz}{1}$; wherein z will be found nearly $= 2.675$. Wherefore x will always be between $5q$ and $2.675q$.

Likewise in the second Equation, if q be supposed $= 1$, then will x be $= 7q$; but if x be supposed infinite, or pretty large in respect to Unity, then $z = \text{Log. } 2 + \text{Log. } \frac{1+z+\frac{1}{2}zz+\frac{1}{6}z^3}{1}$; whence z will be found nearly $= 3.6719$; Wherefore x will be between $7q$ and $3.6719q$.

If these Equations were continued, it would be found that the Limits of z converge continually to the proportion of two to one.

A TABLE of the Limits.

The Value of x will always be

For a single Event, between	$1q$ and $0.693q$.
For a double Event, between	$3q$ and $1.678q$.
For a triple Event, between	$5q$ and $2.675q$.
For a quadruple Event, between	$7q$ and $3.672q$.
For a quintuple Event, between	$9q$ and $4.670q$.
For a sextuple Event, between	$11q$ and $5.668q$.

If the number of Events contended for, as well as the number q be pretty large in respect to Unity; the number of Trials requisite for those Events to Happen n times, will be $\frac{2n-1}{2}q$ or barely nq .

PROBLEM VIII.

Three Gamesters A, B, C, play together on this condition, that he shall win the Set who has soonest got a certain number of Games; the proportion of the Chances which each of them has to get any one Game assigned, or, which is the same thing, the proportion of their Skill, being respectively as a, b, c . Now after they have played some time, they find themselves in this circumstance, that A wants One Game of being up, B Two Games, and C Three; the whole Stake between them being supposed 1: What is the Expectation of each?

G

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SOLUTION.

IN the Circumstance the Gamesters are in, the Set will be ended in Four Games at most; let therefore $a + b + c$ be raised to the fourth Power, and it will be $a^4 + 4a^3b + 6aabb + 4ab^3 + b^4 + 4a^3c + 12aabc + 4b^3c + 6aacc + 12abcc + 6bbcc + 4ac^3 + 12acbb + 4bc^3 + c^4$.

The Terms $a^4 + 4a^3b + 4a^3c + 6aacc + 12aabc + 12abcc$, wherein the Dimensions of a are equal to or greater than the number of Games which A wants, wherein also the Dimensions of b and c are less than the number of Games which B and C want respectively, are intirely Favourable to A , and are part of the Numerator of his Expectation.

In the same manner the Terms $b^4 + 4b^3c + 6bbcc$ are intirely Favourable to B .

And likewise the Terms $4bc^3 + c^4$ are intirely Favourable to C .

The rest of the Terms are common, as Favours partly one of the Gamesters, partly one or both of the other: Wherefore these Terms are so to be divided into their parts, that the parts Favours each Gamester may be added to his Expectation.

Take therefore all the Terms which are common, viz. $6aabb + 4ab^3 + 12abcc + 4ac^3$, and divide them actually into their parts; that is 1°. $6aabb$ into $aabb$, $abab$, $abba$, $baab$, $baba$, $baaa$. Out of these Six parts, one part only, viz. $baaa$ will be found to Favour B , for 'tis only in this Term that two Dimensions of b are placed before one single Dimension of a , and therefore the other Five parts belong to A ; let therefore $5aabb$ be added to the Expectation of A , and $1aabb$ to the Expectation of B . 2° Divide $4ab^3$ into its parts, $abbb$, $babb$, $bbab$, $bbba$. Of these parts there are two belonging to A , and the other two to B ; let therefore $2ab^3$ be added to the Expectation of each. 3° Divide $12abcc$ into its parts; and eight of them will be found Favourable to A , and four to B ; let therefore $8abbc$ be added to the Expectation of A , and $4abbc$ to the Expectation of B . 4° Divide $4ac^3$ into its parts, three of which will be found Favourable to A , and one to C ; Add therefore $3ac^3$ to the Expectation of A , and $1ac^3$ to the Expectation of C . Hence the Numerators of the several Expectations of A , B , C , will be respectively.

1. $a^4 + 4a^3b + 4a^3c + 6a^2cc + 12aabc + 12abcc + 5aabb + 2ab^2 + 8abbc + 3ac^3$.
2. $b^4 + 4b^3c + 6bbcc + 1aabb + 2ab^3 + 4abcc$.
3. $4bc^3 + 1c^4 + 1ac^3$.

The common Denominator of all their Expectations being $a + b + c$.

Therefore if a, b, c are in a proportion of equality, the Odds of winning will be respectively as 57, 18, 6.

If n be the number of all the Games that are wanting, p the number of the Gamesters, a, b, c, d , &c. the proportion of the Chances which each Gamester has respectively to win any one Game assigned; let $a + b + c + d$ &c. be raised to the Power $n + 1 - p$, then proceed as before.

PROBLEM IX.

TWO Gamesters, A and B, each having 12 Counters, play with three Dice, on condition, that if 11 Points come up, B shall give one Counter to A; if 14, A shall give one Counter to B; and that he shall be the winner who shall soonest get all the Counters of his adversary: What are the Probabilities that each of them has of winning?

SOLUTION.

LET the number of Counters which each of them have be $= p$; and let a and b be the number of Chances they have respectively for getting a Counter each cast of the Dice: I say that the Probabilities of winning are respectively as a^p to b^p ; or because in this case $p = 12$, $a = 27$, $b = 15$ as 27^{12} to 15^{12} , or as 9^{12} to 5^{12} , or as 282429536481 to 244140625, which is the proportion assigned by *M. Huygens*, but without any Demonstration:

Or more generally.

Let p be the number of the Counters of A, and q the number of the Counters of B; and let the proportion of the Chances be as a to b . I say that the proportions of the Probabilities which A has to get all the Counters of his adversary will be as $a^q \times a^p - b^p$ to $b^p \times a^q - b^q$.

DEMON-

DEMONSTRATION.

LET it be supposed that *A* has the Counters *E, F, G, H* &c. whose number is *p*, and that *B* has the Counters *I, K, L* &c. whose number is *q*: Moreover let it be supposed that the Counters are the thing play'd for, and that the Value of each of them is to the Value of the following as *a* to *b*, in such a manner that the last Counter of *A* to the first Counter of *B*, be still in that proportion. This being supposed, *A* and *B*, in every circumstance of their Play, may lay down two such Counters as may be proportional to the number of Chances each has to get a single Counter; for in the beginning of the Play *A* may lay down the Counter *H* which is the lowest of his Counters, and *B* the Counter *I* which is his highest; but *H, I* :: *a, b*, therefore *A* and *B* play upon equal Terms. If *A* win of *B*, then *A* may lay down the Counter *I* which he has just got of his adversary, and *B* the Counter *K*; but *I, K* :: *a, b*, therefore *A* and *B* still play upon equal Terms. But if *A* lose the first time, then *A* may lay down the Counter *G*, and *B* the Counter *H*, which he but now got of his adversary; but *G, H* :: *a, b*, and therefore they still Play upon equal Terms as before. So that as long as they Play together, they Play without advantage, or disadvantage, and consequently the Probabilities of winning are reciprocal to the Sums which they expect to win, that is, are proportional to the Sums they respectively have before the Play begins. Whence the Probability which *A* has of winning all the Counters of *B*, is to the Probability which *B* has of winning all the Counters of *A*, as the Sum of the Terms *E, F, G, H* whose number is *p*, to the Sum of the Terms *I, K, L* whose number is *q*; that is, as $a^q \times a^p - b^p$ to $b^p \times a^q - b^q$: As will easily appear if those Terms which are in Geometric Progression are actually summed up by the known methods. Now the Probabilities of winning are not influenced by the supposition here made, of each Counter being to the following in the proportion of *a* to *b*; and therefore when those Counters are supposed of equal Value, or rather of no Value, but serve only to mark the number of Stakes won or lost on either side, the Probabilities of winning will be the same as we have assigned.

R E.

REMARK I.

IF p and q , or either of them are large numbers, 'twill be convenient to work by Logarithms.

Thus, If A and B play a Guinea a Stake, and the number of Chances which A has to win each single Stake be 43, but the number of Chances which B has to win it be 40; and they oblige themselves to play till such time as 100 Stakes are won and lost.

From the Logarithm of 43 = 1.6334685
Subtract the Logarithm of 40 = 1.6020600

Difference = 0.0314085

Multiply this Difference by the number of Stakes to be play'd off, viz. 100; the Product will be 3.1048500, to which answers, in the Tables of Logarithms, the number 1383; wherefore the Odds that A shall win before B are 1383 to 1.

Now in all circumstances wherein A and B venture an equal Sum; the sum of the numbers expressing the Odds, is to their difference, as the Money play'd for, is to the Gain of the one, and the Loss of the other.

Therefore Multiplying 1382, difference of the numbers expressing the Odds, by 100, which is the sum ventured by each Man, and dividing the product by 1384 sum of the numbers expressing the Odds; the Quotient will be 99 Guineas, and about 18^{sh.} — $4\frac{1}{2}^d$, which consequently is to be estimated as the Gain of A .

REMARK II.

IF the number of Stakes which are to be won and lost be unequal, but the number of Chances to win and lose be equal; the Probabilities of winning will be reciprocally proportional to the number of Stakes to be won.

Thus, If A ventures Ten Stakes to win One; the Odds that he wins One before he loses Ten will be as 10 to 1.

But ten Chances to win One, and One Chance to lose ten, makes the Play perfectly equal.

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Therefore he that ventures many Stakes to win but few, has by it neither advantage nor disadvantage.

PROBLEM X.

TWO Gamesters, A and B lay by 24 Counters, and play with Three Dice, on this condition; that if 11 Points come up, A shall take one Counter out of the heap; if 14, B shall take out one, and he shall be reputed to win, who shall soonest get 12 Counters. What are the Probabilities of their winning?

This Problem differs from the preceding in this, that the play will be at an end in 23 Casts of the Dice at most, (that is of those Casts which are favourable either to A or B :) Whereas in the preceding case, the Counters passing continually from one Hand to the other, it will often Happen that A and B will be in some of the same circumstances they were in before, which will make the length of the play unlimited.

SOLUTION.

TAKING a and b in the proportion of the Chances that there are to throw 11 and 14, let $a + b$ be raised to the 23^d Power, that is to such Power as is denoted by the number of all the Counters wanting one: Then shall the 12 first Terms of that Power be to the 12 last in the same proportion as are the respective Probabilities of winning.

PROBLEM XI.

THREE Persons A, B, C out of a heap of 12 Counters, whereof four are White and Eight Black, draw blindfold one Counter at a time in this manner; A begins to draw; B follows A; C follows B; then A begins again; and they continue to draw in the same order, till one of them, who is to be reputed to win, draws the first White one. What are the Probabilities of their winning?

SOLU-

SOLUTION.

LET n be the number of all the Counters, a the number of White ones, b the number of Black ones, and 1 the whole Stake or the sum play'd for.

1° Since A has a Chances for a White Counter, and b Chances for a Black one, it follows that the Probability of his winning is $\frac{a}{a+b}$ or $\frac{a}{n}$; Therefore the Expectation he has upon the Stake 1 arising from the circumstance he is in when he begins to draw is $\frac{a}{n} \times 1 = \frac{a}{n}$. Let it therefore be agreed amongst the adventurers that A shall have no Chance for a White Counter, but that he shall be reputed to have had a Black one, which shall actually be taken out of the heap, and that he shall have the sum $\frac{a}{n}$ paid him out of the Stake for an Equivalent. Now $\frac{a}{n}$ being taken out of the Stake, there will remain $1 - \frac{a}{n} = \frac{n-a}{n} = \frac{b}{n}$.

2° Since B has a Chances for a White Counter, and that the number of remaining Counters is $n-1$, his Probability of winning will be $\frac{a}{n-1}$. Whence his Expectation upon the remaining Stake $\frac{b}{n}$, arising from the circumstance he is now in, will be $\frac{a b}{n \times n-1}$. Suppose it therefore agreed that B shall have the sum $\frac{a b}{n \times n-1}$ paid him out of the Stake, and that a Black Counter be likewise taken out of the heap. This being done, the remaining Stake will be $\frac{b}{n} - \frac{a b}{n \times n-1}$, or $\frac{n b - b - a b}{n \times n-1}$; but $n b - a b = b b$; Wherefore the remaining Stake is $\frac{b \times b - 1}{n \times n-1}$.

3° Since C has a Chances for a White Counter, and that the number of remaining Counters is $n-2$, his Probability of winning will be $\frac{a}{n-2}$: And therefore his Expectation upon the remaining Stake, arising from the circumstance he is now in, will be $\frac{b \times b - 1 \times a}{n \times n-1 \times n-2}$ which we will likewise suppose to be paid him out of the Stake.

4° A may have out of the remainder $\frac{b \times b - 1 \times b - 2 \times a}{n \times n-1 \times n-2 \times n-3}$; and so of the rest till the whole Stake be exhausted.

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Therefore having written the following general Series, *viz.*
 $\frac{a}{n} + \frac{b}{n-1} P + \frac{b-1}{n-2} Q + \frac{b-2}{n-3} R + \frac{b-3}{n-4} S \&c.$ wherein P, Q, R, S &c. denote the preceding Terms, take as many Terms of this Series as there are Units in $b+1$, (for since b represents the number of Black Counters, the number of drawings cannot exceed $b+1$) then take for A the first, fourth, seventh &c. Terms; for B take the second, fifth, eighth &c. Terms; for C the third, sixth &c. and the sums of those Terms will be the respective Expectations of A, B, C ; or because the Stake is fix'd, these sums will be proportional to their respective Probabilities of winning.

Now to apply this to the present case, make $n=12$, $a=4$, $b=8$, and the general Series will become

$\frac{4}{12} + \frac{8}{11} P + \frac{7}{10} Q + \frac{6}{9} R + \frac{5}{8} S + \frac{4}{7} T + \frac{3}{6} V + \frac{2}{5} X + \frac{1}{4} Y$: Or multiplying the whole by 495, to take away the Fractions, the Series will be

$165 + 120 + 84 + 56 + 35 + 20 + 10 + 4 + 1.$

Therefore assign to A $165 + 56 + 10 = 231$; to B $120 + 35 + 4 = 159$; to C $84 + 20 + 1 = 105$, and their Probabilities of winning will be as 231, 159, 105, or as 77, 53, 35.

If there be never so many Gamesters $A, B, C, D \&c.$ whether they take every one of them one Counter or more; or whether the same or a different number of Counters; the Probabilities of winning may be determined by the same general Series.

REMARK I.

THE preceding Series may in any particular case be shorten'd; for if a is $= 1$, then the Series will be

$\frac{1}{n} \times 1 + 1 + 1 + 1 + 1 + 1 + 1 \&c.$

Hence it may be observed, that if the whole number of Counters be exactly divisible by the number of persons concerned in the Play, and that there be but one single White Counter in the whole, there will be no advantage or disadvantage to any one of them from the situation he is in, in respect to the order of drawing.

If

If $a = 2$. then the Series will be

$$\frac{2}{n \times n - 1} \times n - 1 + n - 2 + n - 3 + n - 4 + n - 5 \text{ \&c.}$$

If $a = 3$. then the Series will be

$$\frac{3}{n \times n - 1 \times n - 2} \times n - 1 \times n - 2 + n - 2 \times n - 3 + n - 3 \times n - 4 \text{ \&c.}$$

If $a = 4$. then the Series will be

$$\frac{4}{n \times n - 1 \times n - 2 \times n - 3} \times n - 1 \times n - 2 \times n - 3 + n - 2 \times n - 3 \times n - 4 \text{ \&c.}$$

Wherefore rejecting the common Multipliers; the several Terms of these Series taken in due order will be Proportional to the several Expectations of any number of Gamblers. Thus in the case of this Problem where n is $= 12$, and $a = b$; the Terms of the Series will be

For A.	For B.	For C.
$11 \times 10 \times 9 = 990$	$10 \times 9 \times 8 = 720$	$9 \times 8 \times 7 = 504$
$8 \times 7 \times 6 = 336$	$7 \times 6 \times 5 = 210$	$6 \times 5 \times 4 = 120$
$5 \times 4 \times 3 = 60$	$4 \times 3 \times 2 = 24$	$3 \times 2 \times 1 = 6$
1386	954	630

Hence it follows, that the Probabilities of winning will be respectively as 1386, 954, 630; or dividing all by 18, as 77, 53, 35, as had been before determined.

REMARK II.

BUT if the Terms of the Series are many, it will be convenient to sum them up, by means of the following method, whose Demonstration may be had from the *Methodus Differentialis* of Sir Isaac Newton, printed in his *Analysis*.

Subtract every Term, but the first, from every following Term, and let the remainders be called *first Differences*; subtract in like manner every first difference from the following, and let the remainders be called *second Differences*; subtract again every one of these second differences from that which follows, and call the remainders *third Differences*; and so on, till the last differences become equal. Let the first Term be called a , the second b ; the first of the first differences d' , the

I

first

first of the second differences d'' , the first of the third differences d''' &c. and let the number of Terms which follow the first be x , then will the sum of all those Terms be

$$a + xb + \frac{x}{1} \times \frac{x-1}{2} d' + \frac{x}{1} \times \frac{x-1}{2} \times \frac{x-2}{3} d'' + \frac{x}{1} \times \frac{x-1}{2} \times \frac{x-2}{3} \times \frac{x-3}{4} d''' \text{ \&c.}$$

N. B. If the numbers whose sums are to be taken are the Products of two numbers, the second differences will be equal; if they are the Products of three, the third differences will be equal, and so on. Therefore the number of Terms, which are to be taken after the first, is to exceed only by Unity the number of Factors that enter the composition of every Term.

It may also be observed, that if those numbers are decreasing, it will be convenient to invert their order, and make that the first which was the last.

Thus, supposing the number of all the Counters to be 100, and the number of White ones 4: Then the number of all the Terms belonging to A, B, C will be 97, the last of which $3 \times 2 \times 1$ will belong to A , since 97 being divided by 3, the remainder is 1. Therefore beginning from the lowest Term $3 \times 2 \times 1$, and taking every third Term, as also the differences of those Terms, we shall have the following Scheme

$3 \times 2 \times 1 =$	6			
$6 \times 5 \times 4 =$	120			
$9 \times 8 \times 7 =$	504	384		
$12 \times 11 \times 10 =$	1320	816	432	
$15 \times 14 \times 13 =$	2730	1410	594	162
&c.				

From whence the Values of a, b, d', d'', d''' , in the general Theorem, will be found to be respectively 6, 120, 384, 432, 162; and consequently the sum of all those Terms will be

6 +

$$6 + x \times 120 + \frac{x}{1} \times \frac{x-1}{2} \times 384 + \frac{x}{1} \times \frac{x-1}{2} \times \frac{x-2}{3} \times 432 \\ + \frac{x}{1} \times \frac{x-1}{2} \times \frac{x-2}{3} \times \frac{x-3}{4} \times 162, \text{ or} \\ 6 + 31\frac{1}{2}x + 50\frac{1}{4}xx + 31\frac{1}{2}x^3 + 6\frac{3}{4}x^4, \text{ or} \\ \frac{3}{4} \times \overline{x+1} \times \overline{x+2} \times \overline{3x+1} \times \overline{3x+4}.$$

In like manner it will be found, that the sum of all the Terms which belong to *B*, the last of which is $5 \times 4 \times 3$, is

$$\frac{3}{4} \times \overline{x+1} \times \overline{x+2} \times \overline{3x+5} \times \overline{3x+8}.$$

And also that the sum of all the Terms belonging to *C*, the last of which is $4 \times 3 \times 2$, is

$$\frac{3}{4} \times \overline{x+1} \times \overline{x+2} \times \overline{9xx+27x+16}.$$

Now x in each case represents the number of Terms wanting one, which belong severally to *A*, *B*, *C*; wherefore making $x+1=p$, their several Expectations will be respectively proportional to

$$p \times \overline{p+1} \times \overline{3p-2} \times \overline{3p+1} \\ p \times \overline{p+1} \times \overline{3p+2} \times \overline{3p+5} \\ p \times \overline{p+1} \times \overline{9pp+9p-2}.$$

Again, the number of all the Terms which belong to them all being 97, and *A* being to take first, it follows, that p in the first case is $= 33$, in the other two $= 32$.

Therefore the several Expectations of *A*, *B*, *C* will be respectively proportional to 41225, 39592, 38008.

If the number of all the Counters were 500, and the number of the White ones still 4; then the number of all the Terms representing the Expectations of *A*, *B*, *C* would be 497. Now this number being divided by 3, the Quotient is 165, and the Remainder 2: From whence it follows, that the last Term $3 \times 2 \times 1$ will belong to *B*, the last but one $4 \times 3 \times 2$ to *A*, and the last but two to *C*; it follows also, that for *B*, and *A*, p must be interpreted by 166, but for *C* by 165.

The

The GAME of BASSETTE.

RULES of the PLAT.

THE Dealer, otherwise called the *Banker*, holds a Pack of 52 Cards, and having shuffled them, he turns the whole Pack at once, so as to discover the last Card; after which he lays down by Couples all the Cards.

The Setter, otherwise called the *Ponte*, has 13 Cards in his hand, one of every sort, from the King to the Ace, which 13 Cards are called a *Book*; out of this Book he takes one Card or more at pleasure, upon which he lays a Stake.

The *Ponte* may at his choice, either lay down his Stake before the Pack is turned, or immediately after it is turned; or after any number of Couples are drawn.

The first case being particular shall be calculated by it self; but the other two are comprehended under the same Rules.

Supposing the *Ponte* to lay down his Stake after the Pack is turned, I call 1, 2, 3, 4, 5 &c. the places of those Cards which follow the Card in view, either immediately after the Pack is turned, or after any number of Couples are drawn.

If the Card upon which the *Ponte* has laid a Stake comes out in any odd place, except the first, he wins a Stake equal to his own.

If the Card upon which the *Ponte* has laid a Stake comes out in any even place except the second, he loses his Stake.

If the Card of the *Ponte* comes out in the first place, he neither wins nor loses, but takes his own Stake again.

If the Card of the *Ponte* comes out in the second place, he does not lose his whole Stake, but only a part of it, viz, a half; which to make the calculation more general we will call γ . In this case the *Ponte* is said to be *Faced*.

When the *Ponte* chuses to come in after any number of Couples are down; if his Card happens to be but once in the Pack, and is the very last of all, there is an exception from the general Rule: for tho' it comes out in an odd place which should intitle him to win a Stake equal to his own, yet he neither wins nor loses from that circumstance, but takes back his own Stake.

P R O-

PROBLEM XII.

TO Estimate at Bassete the loss of the Ponte under any circumstances of Cards remaining in the Stock, when he lays his Stake, and of any number of times that his Card is repeated in it.

The Solution of this Problem containing Four Cases, viz. of the Ponts Card being once, twice, three or four times in the Stock; we will give the Solution of all these Cases severally.

SOLUTION of the first Case.

THe Ponte has the following Chances to win or lose, according to the place his Card is in.

1	1	Chance for winning	o
2	1	Chance for losing	y
3	1	Chance for winning	1
4	1	Chance for losing	1
5	1	Chance for winning	1
6	1	Chance for losing	1
*	1	Chance for winning	o

It appears by this Scheme that he has as many Chances to win 1 as to lose 1, and that there are two Chances for neither winning nor losing, viz. the first and last, and therefore that his only Loss is upon account of his being *Faced*: From which 'tis plain that the number of Cards covered by that which is in view being called n , his Loss will be $\frac{y}{n}$, or $\frac{1}{2n}$ supposing $y = \frac{1}{2}$.

SOLUTION of the second Case.

By the first Remark belonging to the XIth Problem it appears that the Chances which the Ponte has to win or lose are proportional to the numbers, $n-1$, $n-2$, $n-3$ &c. Therefore his Chances for winning and losing may be expressed by the following Scheme.

1	$n-1$	Chances for winning	0
2	$n-2$	Chances for losing	y
3	$n-3$	Chances for winning	1
4	$n-4$	Chances for losing	1
5	$n-5$	Chances for winning	1
6	$n-6$	Chances for losing	1
7	$n-7$	Chances for winning	1
8	$n-8$	Chances for losing	1
9	$n-9$	Chances for winning	1
*	1	Chance for losing	1

Now setting aside the first and second number of Chances, it will be found that the difference between the 3^d and 4th is = 1, and that the difference between the 5th and 6th is = 1. The difference between the 7th and 8th also is = 1, and so on. But the number of differences is $\frac{n-3}{2}$, and the sum of all the Chances is $\frac{n}{1} \times \frac{n-1}{2}$. Wherefore the Gain of the Ponte is $\frac{n-3}{n \times n-1}$; but his Loss upon account of the Face is $n-2 \times y$ divided by $\frac{n}{1} \times \frac{n-1}{2}$, or $\frac{2n-4 \times y}{n \times n-1}$: Hence it may be concluded that his Loss upon the whole is

$$\frac{2n-4 \times y - n-3}{n \times n-1}, \text{ or } \frac{1}{n \times n-1} \text{ supposing } y = \frac{1}{2}.$$

That the number of Differences is $\frac{n-3}{2}$ will be made evident from two considerations.

First, the Series $n-3, n-4, n-5$ &c. decreases in Arithmetic Progression, the difference of its Terms being Unity, and the last Term also Unity, therefore the number of its Terms is equal to the first Term $n-3$: But the number of Differences is one half of the number of Terms, therefore the number of Differences will be $\frac{n-3}{2}$.

Secondly, It appears by the XIth Problem, that the number of all the Terms including the two first is always $b+1$; But b in this case is = 2. Therefore the number of all the Terms is $n-1$, from which excluding the two first, the number of remaining Terms will be $n-3$, and consequently the number of Differences will be $\frac{n-3}{2}$.

That the sum of all the Terms is $\frac{n}{1} \times \frac{n-1}{2}$, is evident also from two different considerations. First,

First, In any Arithmetic Progression whereof the first Term is $n-1$, the difference Unity, and the last Term also Unity, the sum of the Progression will be $\frac{n}{1} \times \frac{n-1}{2}$.

Secondly, the Series $\frac{2}{n \times n - 1} \times n-1 + n-2 + n-3$ &c. belonging to the preceding Problem, expresses the sum of the Probabilities of winning, which belong to the several Gamesters in the case of two White Counters, when the number of all the Counters is n . It therefore expresses likewise the sum of the Probabilities of winning, which belong to the Ponte or Banker in the present case: But this sum must always be equal to Unity, it being a certainty that the Ponte or Banker must win; supposing therefore that $n-1, n-2, n-3$ &c. is $= S$. we shall have the Equation $\frac{2S}{n \times n - 1} = 1$. Therefore $S = \frac{n}{1} \times \frac{n-1}{2}$.

SOLUTION of the third Case.

By the first Remark of the XIth Problem it appears that the Chances which the Ponte has to win and lose, may be expressed by the following Scheme.

1	$n-1 \times n-2$ for winning	0
2	$n-2 \times n-3$ for losing	1
3	$n-3 \times n-4$ for winning	1
4	$n-4 \times n-5$ for losing	1
5	$n-5 \times n-6$ for winning	1
6	$n-6 \times n-7$ for losing	1
7	$n-7 \times n-8$ for winning	1
8	$n-8 \times n-9$ for losing	1
*	2×1 for winning	1

Setting aside the first, second and last number of Chances; it will be found that the difference between the 3d and 4th is $2n-8$; the difference between the 5th and 6th $2n-12$, the difference between the 7th and 8th $2n-16$. &c. Now these differences constitute an Arithmetic Progression, whereof the first Term is $2n-8$, the common difference 4, and the last Term 6, being the difference between 4×3 and 3×2 . Wherefore the sum of this Progression is $\frac{n-1}{1} \times \frac{n-5}{2}$, to which adding the last Term 2×1 , which is favourable to the Ponte.

Ponte, the sum total will be $\frac{n-3}{1} \times \frac{n-3}{2}$. But the sum of all the Chances is $\frac{n}{1} \times \frac{n-1}{1} \times \frac{n-2}{3}$; as may be concluded from the first Remark of the preceding Problem: Therefore the Gain of the Ponte is $\frac{3 \times n - 3 \times n - 3}{2 \times n \times n - 1 \times n - 2}$. But his Loss upon account of the Face is $\frac{6 \times n - 2 \times n - 3 \times y}{2 \times n \times n - 1 \times n - 2}$. Consequently his Loss upon the whole will be $\frac{6 \times n - 2 \times n - 3 \times y - 3 \times n - 3 \times n - 3}{2 \times n \times n - 1 \times n - 2}$ or $\frac{3n-9}{2 \times n \times n - 1 \times n - 2}$, Supposing $y = \frac{1}{2}$.

SOLUTION of the fourth Case.

The Chances of the Ponte may be expressed by the following Scheme.

1	$n-1 \times n-2 \times n-3$	for winning	0
2	$n-2 \times n-3 \times n-4$	for losing	y
3	$n-3 \times n-4 \times n-5$	for winning	1
4	$n-4 \times n-5 \times n-6$	for losing	1
5	$n-5 \times n-6 \times n-7$	for winning	1
6	$n-6 \times n-7 \times n-8$	for losing	1
7	$n-7 \times n-8 \times n-9$	for winning	1
*	$3 \times 2 \times 1$	for losing	1

Setting aside the first and second numbers of Chances, and taking the differences between the 3d and 4th, 5th and 6th; 7th and 8th, the last of these differences will be found to be 18. Now if the number of these differences be p , and we begin from the last 18, their sum, from the second Remark of the preceding Problem, will be collected to be $p \times p + 1 \times 4p + 5$: And the number p in this case being $\frac{n-5}{2}$, the sum of these differences will be $\frac{n-5}{2} \times \frac{n-3}{2} \times \frac{2n-5}{1}$. But the sum of all the Chances is $\frac{n}{1} \times \frac{n-1}{1} \times \frac{n-2}{1} \times \frac{n-3}{4}$; wherefore the Gain of the Ponte is $\frac{n-5 \times n-3 \times 2n-5}{n \times n-1 \times n-2 \times n-3}$; now his Loss upon account of the Face is $\frac{n-2 \times n-3 \times n-4 \times 4y}{n \times n-1 \times n-2 \times n-3}$ and therefore his Loss upon the whole is $\frac{4 \times n-2 \times n-4 \times y - n-5 \times 2n-5}{n \times n-1 \times n-2}$ or $\frac{3n-9}{n \times n-1 \times n-2}$, making $y = \frac{1}{2}$. There

There still remains the single Case to be considered, *viz.* what the Loss of the Ponte is, when he lays a Stake before the Pack is turned up ; but there will be no difficulty in it after what we have said, the difference between this Case and the rest being only that he may be Faced by the first Card discovered, which will make his Loss to be $\frac{3n-6}{n \times n - 1 \times n - 3}$, that is, about $\frac{1}{866}$ part of his Stake.

Those who are desirous to try, by a kind of Mechanical Operation, the truth of the Rules which have been given for determining the Loss of the Ponte in any Case, may do it in the following manner. Suppose for Instance it were required to find the Loss of the Ponte when his Card is twice in the Stock, and there are five Cards remaining in the hands of the Banker beside the Card in View. Let them be disposed according to this Scheme.

1,	2,	3,	4,	5
*	*	.	.	.
*	.	*	.	.
*	.	.	*	.
*	.	.	.	*
.	*	*	.	.
.	*	.	*	.
.	*	.	.	*
.	.	*	*	.
.	.	*	.	*
.	.	.	*	*

Where the places filled with Asterisks shew all the Various Positions which the Ponte's Card may obtain ; it is evident that the Ponte has four Chances for neither winning nor losing, three Chances for the Face or for losing $\frac{1}{2}$, two Chances for winning 1, and one Chance for losing 1 ; and consequently that his Loss is $\frac{1}{2}$ to be distributed into 10 parts, the number of all the Chances being 10, which will make his Loss to be $\frac{1}{20}$. Likewise if the number of Cards that are covered by the first were seven, it would be found that the Ponte would have six Chances for neither winning nor losing, five Chances for the Face, four Chances for winning 1, three Chances for losing 1, two Chances again for winning 1, and one Chance for losing 1, which would make his Loss to be $\frac{1}{42}$. And the like may be done for any other Case whatsoever.

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From

From what has been said, a Table may easily be composed, shewing the several Losses of the Ponté in whatever circumstance he may happen to be.

A TABLE for *BASSETTE*.

N	1	2	3	4
52	* * *	* * *	* * *	866
51	* * *	* * *	1735	867
49	98	2352	1602	801
47	94	2162	1474	737
45	90	1980	1351	675
43	86	1806	1234	617
41	82	1640	1122	561
39	78	1482	1015	507
37	74	1332	914	457
35	70	1190	818	409
33	66	1056	727	363
31	62	930	642	321
29	58	812	562	281
27	54	702	487	243
25	50	600	418	209
23	46	506	354	177
21	42	420	295	147
19	38	342	242	121
17	34	272	194	97
15	30	210	151	75
13	26	156	114	57
11	22	110	82	41
9	18	72	56	28
7	14	42	35	17

The Use of this Table will be best explained by one or two Examples.

Exam-

EXAMPLE I.

LET it be proposed to find the Loss of the Ponte when there are 26 Cards remaining in the Stock, and his Card is twice in it.

In the Column *N* find the number 25, which is less by one than the number of Cards remaining in the Stock: Over against it, and under the number 2, which is at the head of the second Column, you will find 600; which is the Denominator of a Fraction whose Numerator is Unity, and which shews that his Loss in that circumstance is one part in six hundred of his Stake.

EXAMPLE II.

TO find the Loss of the Ponte when there is eight Cards remaining in the Stock, and his Card is three times in it.

In the Column *N* find the number 7, less by one than the number of Cards remaining in the Stock: Over against 7, and under the number 3 in the third Column, you will find 35; which denotes that his Loss is one part in thirty five of his Stake.

Corollary I. 'Tis plain from the construction of the Table, that the fewer Cards are in the Stock, the greater is the Loss of the Ponte.

Corollary II. The least Loss of the Ponte, under the same circumstances of Cards remaining in the Stock, is when his Card is but twice in it; the next greater when three times; still greater when four times, but his greatest Loss when 'tis but once.

If the Loss upon the Face were varied, 'tis plain that in all the like circumstances, the Loss of the Ponte would vary accordingly, but it would be easie to compose other Tables to answer that Variation, since the quantity *y*, which has been assumed to represent that Loss may be interpreted at pleasure. For instance, when the Loss upon the Face is $\frac{1}{2}$, it has been found in the Case of 7 Cards covered remaining in the Stock, and the Card of the Ponte being twice in it, that his Loss would be $\frac{1}{42}$, but upon supposition of its being $\frac{2}{3}$, it will be found to be $\frac{4}{63}$.

The

The GAME of PHARAON.

THE Calculation for *Pharaon* is much like the preceding, the reasonings about it being the same; therefore I think it will be sufficient to lay down the Rules of the Play, and the Scheme of the Calculation.

RULES of the PLAY.

First, The Banker holds a Pack of 52 Cards.

Secondly, He draws the Cards one after the other, laying them alternately to his right and left hand.

Thirdly, the Ponte may at his choice set one or more Stakes upon one or more Cards either before the Banker has begun to draw the Cards, or after he has drawn any number of Couples, which are commonly called *Pulls*.

Fourthly, The Banker wins the Stake of the Ponte, when the Card of the Ponte comes out in an odd place on his right hand; but loses as much to the Ponte when it comes out in an even place on his left hand.

Fifthly, The Banker wins half the Ponte's Stake, when in the same Pull the Card of the Ponte comes out twice.

Sixthly, When the Card of the Ponte, being but once in the Stock, happens to be the last, the Ponte neither wins nor loses.

Seventhly, The Card of the Ponte being but twice in the Stock, and the two last Cards happening to be his Cards, he then loses his whole Stake.

PROBLEM XIII.

TO Find at Pharaon the Gain of the Banker, in any Circumstance of Cards remaining in the Stock, and of the number of times that the Ponte's Card is contained in it.

This Problem, containing four Cases, that is, when the Card of the Ponte is once, twice, three or four times in the Stock; we shall give the Solution of these four Cases severally.

SOLU-

SOLUTION of the first Case.

The Banker has the following number of Chances for winning and losing, viz.

1	1	Chance for winning	1
2	1	Chance for losing	1
3	1	Chance for winning	1
4	1	Chance for losing	1
5	1	Chance for winning	1
*	1	Chance for losing	0

Therefore the Gain of the Banker is $\frac{1}{n}$. Supposing n to be the number of Cards in the Stock.

SOLUTION of the second Case.

The Banker has the following Chances for winning and losing, viz.

1	{	$n-2$ Chances for winning	1
		1 Chance for winning	y
2		$n-2$ Chances for losing	1
3	{	$n-4$ Chances for winning	1
		1 Chance for winning	y
4		$n-4$ Chances for losing	1
5	{	$n-6$ Chances for winning	1
		1 Chance for winning	y
6		$n-6$ Chances for losing	1
7	{	$n-8$ Chances for winning	1
		1 Chance for winning	y
8		$n-8$ Chances for losing	1
*		1 Chance for winning	1

Therefore the Gain of the Banker is $\frac{n-2xy+2}{n \times n-1}$, or $\frac{\frac{1}{2}n+1}{n \times n-1}$ supposing $y = \frac{1}{2}$.

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The only thing that deserves to be explained here, is this; how it comes to pass that whereas at *Bassete* the first number of Chances for winning was represented by $n-1$, here 'tis represented by $n-2$. To answer this it must be remember'd, that according to the Law of this Play, if the Ponte's Card comes out in an odd place, the Banker is not thereby entitled to the Ponte's whole Stake: For if it so happens that his Card comes out again immediately after, the Banker wins but one half of it. Therefore the number $n-1$ is divided into two parts $n-2$ and 1, whereof the first is proportional to the Probability which the Banker has for winning the whole Stake of the Ponte; and the second is proportional to the Probability of his winning the half of it.

SOLUTION of the third Case.

The number of Chances which the Banker has for winning, and losing are as follow;

1	{	$n-2 \times n-3$ Chances for winning	1
		$2 \times n-2$ Chances for winning	y
2		$n-2 \times n-3$ Chances for losing	1
3	{	$n-4 \times n-5$ Chances for winning	1
		$2 \times n-4$ Chances for winning	y
4		$n-4 \times n-5$ Chances for losing	1
5	{	$n-6 \times n-7$ Chances for winning	1
		$2 \times n-6$ Chances for winning	y
6		$n-6 \times n-7$ Chances for losing	1
7	{	$n-8 \times n-9$ Chances for winning	1
		$2 \times n-8$ Chances for winning	y
*		2×1 Chances for losing	1

Therefore the Gain of the Banker is $\frac{3y}{2 \times n - 1}$, or $\frac{3}{4 \times n - 1}$ supposing $y = \frac{1}{2}$.

The number of Chances for the Banker to win is divided into two parts, whereof the first expresses the Chances he has for winning the whole Stake of the Ponte, and the second for winning the half thereof.

Now

Now for determining exactly these two parts, it may be considered, that in the first Pull the number of Chances for the first Card to be the Ponte's is $n-1 \times n-2$; also that the number of Chances for the second to be the Ponte's but not the first, is $n-2 \times n-3$: Wherefore the number of Chances for the first to be the Ponte's and not the second, is likewise $n-2 \times n-3$. Hence it follows, that if from the number of Chances for the first Card to be the Ponte's, *viz.* from $n-1 \times n-2$ there be subtracted the number of Chances for the first to be the Ponte's and not the second, *viz.* $n-2 \times n-3$, there will remain the number of Chances for both first and second Cards to be the Ponte's, *viz.* $2 \times n-2$ and so for the rest.

SOLUTION of the fourth Case.

The number of Chances which the Banker has for winning and losing, are as follows;

1	{	$n-2 \times n-3 \times n-4$ Chances for winning	1
		$3 \times n-2 \times n-3$ Chances for winning	y
2		$n-2 \times n-3 \times n-4$ Chances for losing	1
3	{	$n-4 \times n-5 \times n-6$ Chances for winning	1
		$3 \times n-4 \times n-5$ Chances for winning	y
4		$n-4 \times n-5 \times n-6$ Chances for losing	1
5	{	$n-6 \times n-7 \times n-8$ Chances for winning	1
		$3 \times n-6 \times n-7$ Chances for winning	y
6		$n-6 \times n-7 \times n-8$ Chances for losing	1
7	{	$n-8 \times n-9 \times n-10$ Chances for winning	1
		$3 \times n-8 \times n-9$ Chances for winning	y
8		$n-8 \times n-9 \times n-10$ Chances for losing	1
*	{	$2 \times 1 \times 0$ Chances for winning	1
		$3 \times 2 \times 1$ Chances for winning	1
		$2 \times 1 \times 0$ Chances for losing	1

Therefore the Gain of the Banker is $\frac{2n-5}{n-1 \times n-3}y$, or $\frac{2n-5}{2 \times n-1 \times n-3}$ supposing $y = \frac{1}{2}$.

A T A-

A TABLE for *PHARAO N.*

N	1	2	3	4
52	* * *	* * *	* * *	50
50	* * *	94	65	48
48	48	90	62	46
46	46	86	60	44
44	44	82	57	42
42	42	78	54	40
40	40	74	52	38
38	38	70	49	36
36	36	66	46	34
34	34	62	44	32
32	32	58	41	30
30	30	54	38	28
28	28	50	36	26
26	26	46	33	24
24	24	42	30	22
22	22	38	28	20
20	20	34	25	18
18	18	30	22	16
16	16	26	20	14
14	14	22	17	12
12	12	18	14	10
10	10	14	12	8
8	8	11	9	6

The numbers of the foregoing Table, as well as those of the Table for *Bassete*, are sufficiently exact to give at first view an Idea of the advantage of the Banker in all circumstances: But if an absolute degree of exactness be required, it will be easily obtained from the Rules given at the end of each Case.

P R O-

PROBLEM XIV.

IF A, B, C throw in their turns a regular Ball, having four White Faces and eight Black ones; and he be to be reputed to win who shall first bring up one of the White Faces: It is demanded what is the proportion of their respective Probabilities of winning?

SOLUTION.

THe method of reasoning in this Problem is exactly the same with that which we made use of in the Solution of the XIth Problem: But whereas the different throws of the Ball do not diminish the number of its Faces; in the room of the Quantities $b-1$, $b-2$, $b-3$ &c. $n-1$, $n-2$, $n-3$ &c. employed in the Solution of the aforesaid Problem, we must substitute b and n respectively, and the Series belonging to that Problem will be changed into the following, viz.

$$\frac{a}{n} + \frac{ab}{nn} + \frac{abb}{n^3} + \frac{ab^3}{n^4} + \frac{ab^4}{n^5} + \frac{ab^5}{n^6} \text{ \&c.}$$

which is to be continued infinitely: Then taking every third Term thereof, the respective Expectations of A, B, C will be expressed by the three following Series.

$$\begin{aligned} \frac{a}{n} + \frac{ab^3}{n^4} + \frac{ab^6}{n^7} + \frac{ab^9}{n^{10}} + \frac{ab^{12}}{n^{13}} \text{ \&c.} \\ \frac{ab}{nn} + \frac{ab^4}{n^5} + \frac{ab^7}{n^8} + \frac{ab^{10}}{n^{11}} + \frac{ab^{13}}{n^{14}} \text{ \&c.} \\ \frac{abb}{n^3} + \frac{ab^5}{n^6} + \frac{ab^8}{n^9} + \frac{ab^{11}}{n^{12}} + \frac{ab^{14}}{n^{15}} \text{ \&c.} \end{aligned}$$

But the Terms of which each Series is compounded are in Geometric Progression, and the Ratio of each Term to the following the same in each of them; Wherefore the Sums of these Series are in the same proportion as their first Terms, viz. as $\frac{a}{n}$, $\frac{ab}{nn}$, $\frac{abb}{n^3}$ or as nn , bn , bb ; that is, in the present Case, as 144, 96, 64, or as 9, 6, 4. Hence the respective Probabilities of winning will be likewise as the numbers 9, 6, 4.

Corollary I. If there be any other number of Gamesters A, B, C, D &c. playing on the same conditions as above;

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take as many Terms in the Ratio of n to b as there are Gamesters, and those Terms will respectively denote the several Expectations of each Gamester.

Corollary II. If there be any number of Gamesters A, B, C, D &c. playing on the same conditions as above; with this difference only, that all the Faces of the Ball are mark'd by particular Figures, 1, 2, 3, 4 &c. and that a certain number p of those Faces shall intitle A to be the winner; and that likewise any other number of them, as q, r, s, t &c. shall respectively intitle B, C, D, E &c. to be winners: Make $n-p=a, n-q=b, n-r=c, n-s=d, n-t=e$ &c. then in the following Series,

$$\frac{p}{n} + \frac{qa}{nn} + \frac{rab}{n^3} + \frac{sabc}{n^4} + \frac{tabcd}{n^5} \text{ \&c.}$$

the Terms taken in due order shall represent the several Probabilities of winning.

For if the Law of the Play be such, that every Man having once play'd in his turn, shall begin again regularly in the same manner, and that continually till such time as one of them wins: Then take as many Terms of the Series as there are Gamesters, and those Terms shall represent the respective Probabilities of winning.

And if it were the Law of the Play, that every Man should play several times together, for instance twice: Then taking for A the two first Terms, for B the two following, and so on; each Couple of Terms shall represent their respective Probabilities of winning; observing that now p and q are equal, as also r and s .

But if the Law of the Play should be Irregular, then you must take for each Man as many Terms of the Series as will answer that Irregularity, and continue the Series till such time as it gives a sufficient Approximation.

Yet, if at any time the Law of the Play having been Irregular should afterwards recover its Regularity, the Probabilities of winning will (with the help of this Series) be determined by finite expressions.

Thus, if it should be the Law of the Play, that two Men A and B , having play'd irregularly for ten times together, should afterwards play alternately each in his turn: Distribute the ten first Terms of the Series between them, according

ing to their order of playing; and having subtracted the sum of those Terms from Unity, divide the remainder of it between them, in the proportion of the two following Terms, which add respectively to the shares they had before: Then shall the two parts of Unity which *A* and *B* have thus obtained, be proportional to their respective Probabilities of winning.

Of Permutations and Combinations.

Permutations are the Changes which several things can receive in the different Orders in which they may be placed, being considered as taken two and two, three and three, four and four, &c.

Combinations are the various Conjunctions which several things may receive without any respect to Order, being taken two and two, three and three, four and four, &c.

L E M M A.

IF the Probability that an Event shall Happen be $\frac{1}{r}$, and if that Event being supposed to have Happened, the Probability of another's Happening be $\frac{1}{s}$; the Probability of both Happening will be $\frac{1}{r} \times \frac{1}{s}$ or $\frac{1}{rs}$. This having been already Demonstrated in the Introduction, will not require any farther proof.

P R O B L E M XV.

ANT number of Things *a, b, c, d, e, f* being given, out of which Two are taken as it happens: To find the Probability that any one of them, as *a*, shall be the first taken, and any other, as *b*, the second.

S O L U T I O N.

THE number of Things in this Example being Six, it follows that the Probability of taking *a* in the first place is $\frac{1}{6}$: Let *a* be considered as taken, then the Probability of taking *b* will be $\frac{1}{5}$; wherefore the Probability of taking first *a* and then *b* is $\frac{1}{6} \times \frac{1}{5} = \frac{1}{30}$.

Corol-

Corollary. Since the taking of a in the first place and b in the second, is but one single Case of those by which Six Things may change their Order, being taken two and two; it follows, that the number of Changes or Permutations of Six Things taken two and two must be 30.

Generally, Let n be any number of Things; the Probability of taking a in the first place and b in the second, will be $\frac{1}{n \times n-1}$; and the number of Permutations of those Things taken two by two will be $n \times n-1$.

PROBLEM XVI.

ANT number n of Things a, b, c, d, e, f being given, out of which Three are taken as it Happens: To find the Probability that a shall be the first taken, b the second and c the third.

SOLUTION.

THe Probability of taking a in the first place is $\frac{1}{6}$: Let a be considered as taken; the Probability of taking b will be $\frac{1}{5}$: Suppose both a and b taken, the Probability of taking c will be $\frac{1}{4}$. Wherefore the Probability of taking first a , then b , and thirdly c , will be $\frac{1}{6} \times \frac{1}{5} \times \frac{1}{4} = \frac{1}{120}$.

Corollary. Since the taking of a in the first place, b in the second, and c in the third, is but one single Case of those by which Six Things may change their Order, being taken three and three; it follows, that the number of Changes or Permutations of Six Things, taken three and three, must be $6 \times 5 \times 4 = 120$.

Generally, If n be any number of Things; the Probability of taking a in the first place, b in the second and c in the third will be $\frac{1}{n} \times \frac{1}{n-1} \times \frac{1}{n-2}$. And the number of Permutations of three Things will be $n \times n-1 \times n-2$.

General COROLLARY.

The number of Permutations of n Things, out of which as many are taken together as there are Units in p , will be $n \times n-1 \times n-2 \times n-3$, &c. continued to so many Terms as there are Units in p .

Thus,

Thus, the number of Permutations of Six Things taken four and four, will be $6 \times 5 \times 4 \times 3 = 360$. Likewise the number of Permutations of Six Things taken all together will be $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$.

PROBLEM XVII.

TO Find the Probability that any number of Things, whereof some are repeated several times, shall all be taken in any Order proposed: For Instance, that aabbcccc shall be taken in the Order wherein they are written.

SOLUTION.

THe Probability of taking *a* in the first place is $\frac{2}{9}$: Supposing one *a* to be taken; the Probability of taking the other is $\frac{1}{8}$. Let now the two first Letters be supposed to be taken, the Probability of taking *b* will be $\frac{3}{7}$: Let this also be supposed taken, the Probability of taking another *b* will be $\frac{2}{6}$: Let this likewise be supposed taken, the Probability of taking the third *b* will be $\frac{1}{5}$; after which there remaining nothing but the Letter *c*, the Probability of taking it becomes a certainty, and consequently is equal to Unity. Wherefore the Probability of taking all those Letters in the Order given is $\frac{2}{9} \times \frac{1}{8} \times \frac{3}{7} \times \frac{2}{6} \times \frac{1}{5}$.

Corollary. Therefore the number of Permutations which the Letters aabbcccc may receive, being taken all together will be $\frac{9 \times 8 \times 7 \times 6 \times 5}{2 \times 1 \times 3 \times 2 \times 1}$.

Generally. The number of Permutations which any number *n* of Things may receive, being taken all together, whereof the first sort is repeated *p* times, the second *q* times, the third *r* times, the fourth *s* times, &c. will be the Series $n \times n-1 \times n-2 \times n-3 \times n-4$, &c. continued to so many Terms as there are Units in $p+q+r$ or $n-s$, divided by the Product of the following Series, viz. $p \times p-1 \times p-2$, &c. $\times q \times q-1 \times q-2$, &c. $\times r \times r-1 \times r-2$, &c. whereof the first must be continued to so many Terms as there are Units in *p*; the second, to so many Terms as there are Units in *q*; the third, to so many Terms as there are Units in *r* &c.

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PROBLEM XVIII.

ANY number of Things a, b, c, d, e, f being given: To find the Probability that, in taking two of them as it may Happen, both a and b shall be taken independently, or without any regard to Order.

SOLUTION.

THE Probability of taking a or b in the first place will be $\frac{2}{6}$; suppose one of them taken, as for Instance a , then the Probability of taking b will be $\frac{1}{5}$. Wherefore the Probability of taking both a and b will be $\frac{2}{6} \times \frac{1}{5} = \frac{2}{30} = \frac{1}{15}$.

Corollary. The taking of both a and b is but one single Case of all those by which Six Things may be combined two and two; wherefore the number of Combinations of Six Things taken two and two will be $\frac{6}{1} \times \frac{5}{2} = 15$.

Generally. The number of Combinations of n Things, taken two and two, will be $\frac{n}{1} \times \frac{n-1}{2}$.

PROBLEM XIX.

ANY number of Things a, b, c, d, e, f being given: To find the Probability, that in taking three of them as it Happens, they shall be any three proposed, as a, b, c ; no respect being had to Order.

SOLUTION.

THE Probability of taking either a , or b , or c in the first place will be $\frac{3}{6}$. Suppose one of them as a to be taken, then the Probability of taking b , or c in the second place will be $\frac{2}{5}$. Again let either of them taken, as suppose b ; then the Probability of taking c in the third place will be $\frac{1}{4}$; wherefore the Probability of taking the three Things proposed, viz. a, b, c will be $\frac{3}{6} \times \frac{2}{5} \times \frac{1}{4}$.

Corol.

Corollary. The taking of a, b, c is but one single Case of all those by which Six Things may be combined three and three; wherefore the number of Combinations of Six Things taken three and three will be $\frac{6}{1} \times \frac{5}{2} \times \frac{4}{3} = 20$.

Generally. The number of Combinations of n Things combined according to the number p , will be

$\frac{n \times n-1 \times n-2 \times n-3 \times n-4}{p \times p-1 \times p-2 \times p-3 \times p-4}$ &c. Both Numerator and Denominator being continued to so many Terms as there are Units in p .

PROBLEM XX.

TO find what Probability there is, that in taking as it Happens Seven Counters out of Twelve, whereof four are White and eight Black, three of them shall be White ones.

SOLUTION.

First, Find the number of Chances for taking three White ones out of four, which will be $\frac{4}{1} \times \frac{3}{2} \times \frac{2}{3} = 4$.

Secondly, Find the number of Chances for taking four Black ones out of eight: These Chances will be found to be $\frac{8 \times 7 \times 6 \times 5}{1 \times 2 \times 3 \times 4} = 70$.

Thirdly, Because every one of the preceding Chances may be joined with every one of the latter, it follows, that the number of Chances for taking three White ones and four Black ones, will be $4 \times 70 = 280$.

Fourthly, Find the number of Chances for taking four White ones out of four, which will be found to be $\frac{4}{1} \times \frac{3}{2} \times \frac{2}{3} \times \frac{1}{4} = 1$.

Fifthly, Find the number of Chances for taking three Black ones out of eight, which will be $\frac{8}{1} \times \frac{7}{2} \times \frac{6}{3} = 56$.

Sixthly, Multiply these two last numbers together, and the Product 56 will shew that there are 56 Chances for taking four White ones and three Black ones; which is a Case not expressed in the Problem, yet is implied: For he who undertakes to take three White Counters out of eight, is reputed to be a winner tho' he takes four; unless the contrary be expressly stipulated.

Seventhly,

Seventhly, Wherefore the number of Chances for taking three White Counters will be $280 + 56 = 336$.

Eighthly, Seek the number of all the Chances for taking seven Counters out of twelve, which will be found to be

$$\frac{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6}{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = 792.$$

Lastly, Divide the preceding number 336 by the last 792, and the Quotient $\frac{336}{792}$, or $\frac{14}{33}$ will be the Probability required.

Corollary. Let n be the number of all the Counters, a the number of White ones, b the number of Black ones, c the number of Counters to be taken out of the number n ; then the number of Chances for taking none of the White ones, or one single White, or two White ones and no more, or three White ones and no more, or four White ones and no more, &c. will be exprest as follows.

$$\frac{b}{1} \times \frac{b-1}{2} \times \frac{b-2}{3} \times \frac{b-3}{4} \text{ \&c. } \times \frac{a}{1} \times \frac{a-1}{2} \times \frac{a-2}{3} \times \frac{a-3}{4} \text{ \&c.}$$

The number of Terms wherein b enters being always equal to $c-a$, and the whole number of Terms equal to c .

But the number of all the Chances for taking a certain number c of Counters out of the number n , with one or more White ones, or without any, will be

$$\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{n-5}{6} \times \frac{n-6}{7} \times \frac{n-7}{8} \text{ \&c.}$$

which Series must be continued to so many Terms as there are Units in c .

R E M A R K.

IF the numbers n and c were large, such as 40000 and 8000, the foregoing method would seem impracticable, by reason of the vast number of Terms to be taken in both Series, whereof the first is to be divided by the second: Tho' if those Terms were actually set down, a great many of them being common Divisors, might be expunged out of both Series. However to avoid the trouble of setting down so many Terms, it will be convenient to use the following Theorem, which is a contraction of that Method.

Let therefore n be the number of all the Counters, a the number of White ones, c the number of Counters to be taken out

out of the number n , p the number of White Counters to be taken precisely in the number c : Then making $n-c=d$. I say that the Probability of taking precisely the number p of White Counters will be

$$\frac{c \times c-1 \times c-2 \times c-3 \times c-4 \times c-5 \times c-6 \times c-7 \times c-8 \times c-9 \times c-10 \times c-11 \times c-12 \times c-13 \times c-14 \times c-15 \times c-16 \times c-17 \times c-18 \times c-19 \times c-20 \times c-21 \times c-22 \times c-23 \times c-24 \times c-25 \times c-26 \times c-27 \times c-28 \times c-29 \times c-30 \times c-31 \times c-32 \times c-33 \times c-34 \times c-35 \times c-36 \times c-37 \times c-38 \times c-39 \times c-40 \times c-41 \times c-42 \times c-43 \times c-44 \times c-45 \times c-46 \times c-47 \times c-48 \times c-49 \times c-50 \times c-51 \times c-52 \times c-53 \times c-54 \times c-55 \times c-56 \times c-57 \times c-58 \times c-59 \times c-60 \times c-61 \times c-62 \times c-63 \times c-64 \times c-65 \times c-66 \times c-67 \times c-68 \times c-69 \times c-70 \times c-71 \times c-72 \times c-73 \times c-74 \times c-75 \times c-76 \times c-77 \times c-78 \times c-79 \times c-80 \times c-81 \times c-82 \times c-83 \times c-84 \times c-85 \times c-86 \times c-87 \times c-88 \times c-89 \times c-90 \times c-91 \times c-92 \times c-93 \times c-94 \times c-95 \times c-96 \times c-97 \times c-98 \times c-99 \times c-100}{n \times n-1 \times n-2 \times n-3 \times n-4 \times n-5 \times n-6 \times n-7 \times n-8 \times n-9 \times n-10 \times n-11 \times n-12 \times n-13 \times n-14 \times n-15 \times n-16 \times n-17 \times n-18 \times n-19 \times n-20 \times n-21 \times n-22 \times n-23 \times n-24 \times n-25 \times n-26 \times n-27 \times n-28 \times n-29 \times n-30 \times n-31 \times n-32 \times n-33 \times n-34 \times n-35 \times n-36 \times n-37 \times n-38 \times n-39 \times n-40 \times n-41 \times n-42 \times n-43 \times n-44 \times n-45 \times n-46 \times n-47 \times n-48 \times n-49 \times n-50 \times n-51 \times n-52 \times n-53 \times n-54 \times n-55 \times n-56 \times n-57 \times n-58 \times n-59 \times n-60 \times n-61 \times n-62 \times n-63 \times n-64 \times n-65 \times n-66 \times n-67 \times n-68 \times n-69 \times n-70 \times n-71 \times n-72 \times n-73 \times n-74 \times n-75 \times n-76 \times n-77 \times n-78 \times n-79 \times n-80 \times n-81 \times n-82 \times n-83 \times n-84 \times n-85 \times n-86 \times n-87 \times n-88 \times n-89 \times n-90 \times n-91 \times n-92 \times n-93 \times n-94 \times n-95 \times n-96 \times n-97 \times n-98 \times n-99 \times n-100} \times c \times c-1 \times c-2 \times c-3 \times c-4 \times c-5 \times c-6 \times c-7 \times c-8 \times c-9 \times c-10 \times c-11 \times c-12 \times c-13 \times c-14 \times c-15 \times c-16 \times c-17 \times c-18 \times c-19 \times c-20 \times c-21 \times c-22 \times c-23 \times c-24 \times c-25 \times c-26 \times c-27 \times c-28 \times c-29 \times c-30 \times c-31 \times c-32 \times c-33 \times c-34 \times c-35 \times c-36 \times c-37 \times c-38 \times c-39 \times c-40 \times c-41 \times c-42 \times c-43 \times c-44 \times c-45 \times c-46 \times c-47 \times c-48 \times c-49 \times c-50 \times c-51 \times c-52 \times c-53 \times c-54 \times c-55 \times c-56 \times c-57 \times c-58 \times c-59 \times c-60 \times c-61 \times c-62 \times c-63 \times c-64 \times c-65 \times c-66 \times c-67 \times c-68 \times c-69 \times c-70 \times c-71 \times c-72 \times c-73 \times c-74 \times c-75 \times c-76 \times c-77 \times c-78 \times c-79 \times c-80 \times c-81 \times c-82 \times c-83 \times c-84 \times c-85 \times c-86 \times c-87 \times c-88 \times c-89 \times c-90 \times c-91 \times c-92 \times c-93 \times c-94 \times c-95 \times c-96 \times c-97 \times c-98 \times c-99 \times c-100$$

Here it is to be observed, that the Numerator consists of three Series, which are to be Multiplied together; whereof the first contains as many Terms as there are Units in p , the second as many as there are Units in $a-p$, the third as many as there are Units p : And the Denominator as many as there are Units in a .

PROBLEM XXI.

IN A Lottery consisting of 40000 Tickets, among which are Three particular Benefits: What is the Probability that taking 8000 of them, one or more of the Three particular Benefits shall be amongst them?

SOLUTION.

First in the Theorem belonging to the Remark of the foregoing Problem, having substituted respectively, 8000, 40000, 32000, 3 and 1, in the room of c , n , d , a and p ; it will appear, that the Probability of taking precisely one of the Three particular Benefits will be

$$\frac{8000 \times 32000 \times 31999 \times 3}{40000 \times 39999 \times 39998}, \text{ or } \frac{48}{125} \text{ nearly.}$$

Secondly, c , n , d , a being interpreted as before, let us suppose $p = 2$. Hence the Probability of taking precisely Two of the particular Benefits will be found to be

$$\frac{8000 \times 7999 \times 32000 \times 3}{40000 \times 39999 \times 39998}, \text{ or } \frac{12}{125} \text{ very near.}$$

Thirdly, Making $p = 3$. The Probability of taking all the Three particular Benefits will be found to be

$$\frac{8000 \times 7999 \times 7998}{40000 \times 39998 \times 39997}, \text{ or } \frac{1}{125} \text{ very near.}$$

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Wherefore the Probability of taking one or more of the Three particular Benefits will be $\frac{48+12+1}{125}$, or $\frac{61}{125}$ very near.

N. B. These three Operations might be contracted into one, by inquiring what the Probability is, that none of the particular Benefits may be taken; for then it will be found to be $\frac{32000 \times 31999 \times 31998}{40000 \times 39999 \times 39998} = \frac{64}{125}$ nearly; which being subtracted from 1, the Remainder $1 - \frac{64}{125}$, or $\frac{61}{125}$ will shew the Probability required.

PROBLEM XXII.

TO Find how many Tickets ought to be taken in a Lottery consisting of 40000, among which there are Three particular Benefits, to make it as Probable that one or more of those Three may be taken as not.

SOLUTION.

LET the number of Tickets requisite to be taken be x : It will follow therefore from the Theorem belonging to the Remark of the XXth. Problem, that the Probability of not taking amongst them any of the particular Benefits will be $\frac{n-x}{n} \times \frac{n-x-1}{n-1} \times \frac{n-x-2}{n-2}$. But this Probability is $= \frac{1}{2}$, since the Probability of the contrary is $\frac{1}{2}$ by Hypothesis; whence it follows that $\frac{n-x}{1} \times \frac{n-x-1}{2} \times \frac{n-x-2}{3} = \frac{1}{2}$. This Equation being solved, the value of x will be found to be nearly 8252.

N. B. The Factors, whereof both Numerator and Denominator are composed being in Arithmetic Progression, and the difference being very small in respect of n ; those Terms may be considered as being in Geometric Progression, wherefore the Cube of the middle Term $\frac{n-x-1}{n-1}$ may be supposed equal to the Product of those Terms; from whence will arise the Equation $\frac{(n-x-1)^3}{(n-1)^3} = \frac{1}{2}$ or $\frac{(n-x)^3}{n^3} = \frac{1}{2}$ (neglecting Unity in both Numerator and Denominator) and con-

consequently x will be found to be $n \times 1 - \sqrt[3]{\frac{1}{n}}$, but n is $= 40000$, and $1 - \sqrt[3]{\frac{1}{n}} = 0.2063$; Therefore $x = 8252$.

In the Remark belonging to the Vth Problem, a Rule was given for finding the number of Tickets that were to be taken to make it as Probable that one or more of the Benefits should be taken, as not; but in that Rule it is supposed that the proportion of the Blanks to the Prizes was often repeated, as it usually is in Lotteries: Now in the Case of the present Problem, the particular Benefits being but Three in all, the remaining Tickets are to be considered as Blanks in respect of them; from whence it follows, that the proportion of the number of Blanks to one Prize being very near as 13332 to 1, and that proportion being repeated but three times in the whole number of Tickets, the Rule there given would not have been sufficiently exact in this Case; to supply which it was thought necessary to give the Solution of this Problem.

PROBLEM XXIII.

TO Find at Pharaon, how much it is that the Banker gets per Cent of all the Money that is adventured,

HYPOTHESIS.

I Suppose, *First*, that there is but one single Ponte: *Secondly*, That he lays his Money upon one single Card at a time: *Thirdly*, That he begins to take a Card in the beginning of the Game: *Fourthly*, That he continues to take a new Card after the laying down of every Pull: *Fifthly*, That when there remains but Six Cards in the Stock, he ceases to take a Card.

SOLUTION.

WHEN at any time the Ponte lays a new Stake upon a Card taken as it Happens out of his Book, let the number of Cards that are already laid down by the Banker be supposed equal to x .

Now

Now in this circumstance, the Card taken by the Ponte has either past four times, or three times, or twice, or once, or not at all.

First, If it has passed four times, he can be no loser upon that account.

Secondly, If it has passed three times, then his Card is once in the Stock; now the number of Cards remaining in the Stock being $n-x$, it follows by the first Case of the XIIIth Problem that the loss of the Ponte will be $\frac{1}{n-x}$: But by the Remark belonging to the XXth Problem, the Probability that his Card has passed three times precisely in x Card, is $\frac{x \times x - 1 \times x - 2 \times n - x \times 4}{n \times n - 1 \times n - 2 \times n - 3}$. Now supposing the Denominator e-

qual to s , Multiply the loss he will suffer (if he has that Chance) by the Probability of having it, and the Product $\frac{x \times x - 1 \times x - 2 \times 4}{s}$, will be his absolute loss in that circumstance.

Thirdly, If it has passed twice, his loss by the second Case of the XIIIth Problem will be $\frac{\frac{1}{2}n - \frac{1}{2}x + 1}{n - x \times n - x - 1}$, but the Probability that his Card has passed twice in x Cards, is by the Remark of the XXth Problem, $\frac{x \times x - 1 \times n - x \times n - x - 1 \times 6}{s}$; wherefore Multiplying the loss he will suffer (if he has that Chance) by the Probability of his having it, the Product $\frac{x \times x - 1 \times \frac{1}{2}n - \frac{1}{2}x + 1 \times 6}{s}$ will be his absolute loss in that circumstance.

Fourthly, If it has passed once, his loss Multiplied into the Probability that it has passed, will make his absolute loss to be $\frac{x \times n - x \times n - x - 2 \times 3}{s}$.

Fifthly, If it has not yet passed, his loss Multiplied into the Probability that it has not passed, will make his absolute loss to be $\frac{n - x \times n - x - 2 \times 2n - 2x - 5}{s}$.

Now the Sum of all these losses of the Ponte's will be $\frac{n^3 - \frac{2}{3}nn + 5n - 3x - \frac{7}{2}xx + 3x^2}{s}$, and this is the loss he suffers by venturing a new Stake after any number of Cards x are past.

But

But the number of Pulls which at any time are laid down, is always one half of the number of Cards that are past; wherefore calling t the number of those Pulls, the Loss of the Ponte may be expressed thus, $\frac{n^3 - \frac{2}{3}nn + 5n - 6t - 6tt + 24t^3}{5}$.

Let now p be the number of Stakes which the Ponte adventures; let also the Loss of the Ponte be divided into two parts, viz. $\frac{n^3 - \frac{2}{3}nn + 5n}{5}$, and $\frac{-6t - 6tt + 24t^3}{5}$.

And since he adventures a Stake p times; it follows, that the first part of his Loss will be $\frac{pn^3 - \frac{2}{3}pnn + 5pn}{5}$.

In order to find the second part, let t be interpreted successively by 0, 1, 2, 3 &c. to the last Term $p-1$; Then in the room of $6t$ we shall have a sum of numbers in Arithmetic Progression to be Multiplied by 6; in the room of $6tt$ we shall have a sum of Squares whose Roots are in Arithmetic Progression to be Multiplied by 6; and in the room of $24t^3$ we shall have a sum of Cubes whose Roots are in Arithmetic Progression to be Multiplied by 24: These several sums being collected, according to the II^d Remark on the XIth Problem, will be found to be $\frac{6p^4 - 14p^3 + 6pp + 2p}{5}$, and therefore the whole Loss of the Ponte will be

$$\frac{pn^3 - \frac{2}{3}pnn + 5pn + 6p^4 - 14p^3 + 6pp + 2p}{5}.$$

Now this being the Loss which the Ponte sustains by adventuring the sum p , each Stake being supposed equal to Unity, it follows, that the Loss *per Cent* of the Ponte, or the Gain *per*

Cent of the Banker is $\frac{n^3 - \frac{2}{3}nn + 5n + 6p^4 - 14pp + 6p + 2}{5} \times 100$,

or $\frac{2n-5}{2 \times n - 1 \times n - 3} + \frac{p-1 \times 6pp - 8p - 2}{n \times n - 1 \times n - 2 \times n - 3} \times 100$. Let now n be interpreted by 52, and p by 23; and the Gain of the Banker will be found to be 2.99251, that is 2 *l.* 19^{sh.} 10d. *per Cent.*

By the same Method of arguing, it will be found that the Gain *per Cent* of the Banker, at *Bassete*, will be

$\frac{3n-9}{n \times n - 1 \times n - 2} + \frac{4p \times p - 1 \times p - 2}{n \times n - 1 \times n - 2 \times n - 3} \times 100$. Let n be interpreted by 51, and p by 23; and the foregoing expression will

will become 0.790582, or 15^{sh.} 9 d. half-penny. The consideration of the first Stake, which is adventured before the Pack is turned, being here omitted as being out of the general Rule: But if that Case be taken in, and the Ponte adventures 100*l.* in 24 Stakes, the Gain of the Banker will be diminished, and becomes only 0.76245, that is, 15^{sh.} 3 d. very near: And this is to be estimated, as the gain *per Cent* of the Banker when he takes but half Face.

Now whether the Ponte takes one Card at a time or several Cards, the Gain *per Cent* of the Banker continues the same: Whether the Ponte keeps constantly to the same Stake, or some times doubles or triples it, the Gain *per Cent* is still the same: Whether there be but one single Ponte or several, his Gain *per Cent* is not thereby altered. Wherefore the Gain *per Cent* of the Banker of all the Money that is adventured at *Pharaon* is 2 *l.* 19^{sh.} 10 d. and at *Bassete* 15^{sh.} 3 d.

PROBLEM XXIV.

Supposing A and B to play together, the Chances they have respectively to win being as a to b, and B obliging himself to Set to A, so long as A wins without interruption: What is the Advantage that A gets by his Hand?

SOLUTION.

First, If *A* and *B* each Stake One, the Gain of *A* on the first Game is $\frac{a-b}{a+b}$.

Secondly, His Gain on the second Game will also be $\frac{a-b}{a+b}$, provided he should happen to win the first: But the Probability of *A's* winning the first Game is $\frac{a}{a+b}$. Wherefore his Gain on the second Game will be $\frac{a}{a+b} \times \frac{a-b}{a+b}$.

Thirdly, His Gain on the third Game, after winning the two first, will be likewise $\frac{a-b}{a+b}$: But the Probability of *A's* winning the two first Games is $\frac{a^2}{a+b^2}$; Wherefore his Gain on the third

third Game, when it is estimated before the Play begins, is $\frac{aa}{a+b} \times \frac{a-b}{a+b}$ &c.

Fourthly, Wherefore the Gain of the Hand of *A* is an infinite Series, viz. $1 + \frac{a}{a+b} + \frac{aa}{(a+b)^2} + \frac{a^3}{(a+b)^3} + \frac{a^4}{(a+b)^4}$ &c. to be Multiplied by $\frac{a-b}{a+b}$. But the sum of that infinite Series is $\frac{a+b}{b}$; Wherefore the Gain of the Hand of *A* is $\frac{a+b}{b} \times \frac{a-b}{a+b} = \frac{a-b}{b}$.

Corollary I. If *A* has the advantage of the Odds, and *B* Sets his Hand out, the Gain of *A* is the difference of the numbers expressing the Odds divided by the lesser. Thus if *A* has the Odds of Five to Three, then his Gain will be $\frac{5-3}{3} = \frac{2}{3}$.

Corollary II. If *B* has the Disadvantage of the Odds, and *A* Sets his Hand out, the Loss of *B* will be the difference of the number expressing the Odds divided by the greater: Thus if *B* has but Three to Five of the Game, his Loss will be $\frac{2}{5}$.

Corollary III. If *A* and *B* do mutually engage to Set to one-another as long as either of them wins without interruption, the Gain of *A* will be found to be $\frac{aa-bb}{ab}$: That is the sum of the numbers expressing the Odds Multiplied by their difference, the product of that Multiplication being divided by the Product of the numbers expressing the Odds. Thus if the Odds were as Five to Three, the sum of 5 and 3 is 8, and the difference 2; Multiply 8 by 2, and the Product 16 being divided by 15 (Product of the number expressing the Odds) the Quotient will be $\frac{16}{15}$, or $1\frac{1}{15}$, which therefore will be the Gain of *A*.

PROBLEM XXV.

AN T given number of Letters *a, b, c, d, e, f* &c. all of them different, being taken promiscuously, as it Happens: To find the Probability that some of them shall be found in their places, according

according to the rank they obtain in the Alphabet; and that others of them shall at the same time be found out of their places.

SOLUTION.

LET the number of all the Letters be $= n$; let the number of those that are to be in their places be $= p$, and the number of those that are to be out of their places $= q$. Suppose for Brevity sake $\frac{1}{n} = r$, $\frac{1}{n \times n - 1} = s$, $\frac{1}{n \times n - 1 \times n - 2} = t$, $\frac{1}{n \times n - 1 \times n - 2 \times n - 3} = v$ &c. then let all the Quantities r, s, t, v &c. be written down with Signs alternately positive and negative, beginning at r , if p be $= 0$; at s , if $p = 1$; at t , if $p = 2$ &c. Prefix to these Quantities the respective Coefficients of a Binomial Power, whose Index is $= q$: This being done, those Quantities taken all together will express the Probability required; thus the Probability that in Six Letters taken promiscuously, two of them, viz. a and b shall be in their places, and three of them, viz. c, d, e out of their places, will be

$$\frac{1}{6 \times 5} - \frac{3}{6 \times 5 \times 4} + \frac{3}{6 \times 5 \times 4 \times 3} - \frac{1}{6 \times 5 \times 4 \times 3 \times 2} = \frac{11}{720}$$

And the Probability that a shall be in its place, and b, c, d, e out of their places, will be

$$\frac{1}{6} - \frac{4}{6 \times 5} + \frac{6}{6 \times 5 \times 4} - \frac{4}{6 \times 5 \times 4 \times 3} + \frac{1}{6 \times 5 \times 4 \times 3 \times 2} = \frac{53}{720}$$

The Probability that a shall be in its place, and b, c, d, e, f out of their places, will be

$$\frac{1}{6} - \frac{5}{6 \times 5} + \frac{10}{6 \times 5 \times 4} - \frac{10}{6 \times 5 \times 4 \times 3} + \frac{5}{6 \times 5 \times 4 \times 3 \times 2} - \frac{1}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{44}{720}, \text{ or } \frac{11}{180}$$

The Probability that a, b, c, d, e, f shall be all displaced is,

$$1 - \frac{6}{6} + \frac{15}{6 \times 5} - \frac{20}{6 \times 5 \times 4} + \frac{15}{6 \times 5 \times 4 \times 3} - \frac{6}{6 \times 5 \times 4 \times 3 \times 2} + \frac{1}{6 \times 5 \times 4 \times 3 \times 2 \times 1}, \text{ or } 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} = \frac{265}{720} = \frac{53}{144}$$

Hence

Hence it may be concluded that the Probability that one or more of them will be found in their places is $1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720} = \frac{91}{144}$; and that the Odds that one or more of them will be so found are as 91 to 53.

N. B. So many Terms of this last Series are to be taken as there are Units in n .

DEMONSTRATION.

THE number of Chances for the Letter a to be in the first place contains the number of Chances, by which a being in the first place, b may be in the second, or out of it: This is an Axiom of common Sense, of the same degree of Evidence as that the Whole is equal to all its Parts.

From this it follows, that if from the number of Chances that there are for a to be in the first place, there be subtracted the number of Chances that there are for a to be in the first place, and b at the same time in the second, there will remain the number of Chances, by which a being in the first place, b may be excluded the second.

For the same reason it follows, that if from the number of Chances that there are for a and b to be respectively in the first and second places, there be subtracted the number of Chances by which a , b and c may be respectively in the first, second and third places; there will remain the number of Chances by which a being in the first and b in the second, c may be excluded the third place: And so of the rest.

Let $+ a'$ denote the Probability that a shall be in the first place, and let $- a'$ denote the Probability of its being out of it. Likewise let the Probabilities that b shall be in the second place or out of it be respectively expressed by $+ b''$ and $- b''$.

Let the Probability that, a being in the first place, b shall be in the second, be expressed by $a' + b''$: Likewise let the Probability that a being in the first place, b shall be excluded the second, be expressed by $a' - b''$.

Generally. Let the Probability there is, that as many as are to be in their proper places, shall be so, and at the same time that as many others as are to be out of their proper places

R

shall

shall be so found, be denoted by the particular Probabilities of their being in their proper places, or out of them, written all together: So that for Instance $a' + b'' + c''' - d'''' - e'''''$ may denote the Probability that a, b and c shall be in their proper places, and that at the same time both d and e shall be excluded their proper places.

Now to be able to derive a proper conclusion by vertue of this Notation, it is to be observed, that of the Quantities which are here considered, those from which the Subtraction is to be made, are indifferently composed of any number of Terms connected by $+$ and $-$; the Quantities which are to be subtracted do exceed by one Term those from which the subtraction is to be made; the rest of the Terms being alike and their signs alike: And the remainder will contain all the Quantities that are alike with their own signs, and also the Quantity Exceeding, but with its sign varied.

It having been demonstrated in what we have said of Permutations and Combinations, that $d' = \frac{1}{n}$, $d' + b'' = \frac{1}{n \times n - 1}$, $d' + b'' + c''' = \frac{1}{n \times n - 1 \times n - 2}$, let $\frac{1}{n}$, $\frac{1}{n \times n - 1}$ &c. be respectively called r, s, t, v &c. This being supposed, we may come to the following conclusions.

$$\begin{array}{l}
 b'' = r \\
 b'' + a' = s \\
 \text{Therefore } b'' - a' = r - s \\
 c''' + b'' = s \quad \text{for the same reason that } a' + b'' = s \\
 c''' + b'' + a' = t \\
 2^{\circ} \text{ Theref. } c''' + b'' - a' = s - t \\
 c''' - a' = r - s \quad \text{By the first Conclusion.} \\
 c''' - a' + b'' = s - t \quad \text{By the 2d.} \\
 3^{\circ} \text{ Theref. } c''' - a' - b'' = r - 2s + t \\
 d'''' + c''' + b'' = t \\
 d'''' + c''' + b'' + a' = v \\
 4^{\circ} \text{ Theref. } d'''' + c''' + b'' - a' = t - v \\
 d'''' + c''' - a' = s - t \quad \text{By the 2d. Conclusion.} \\
 d'''' + c''' - a' + b'' = t - v \quad \text{By the 4th.} \\
 5^{\circ} \text{ Theref. } d'''' + c''' - a' - b'' = s - 2t + v
 \end{array}$$

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$$\begin{array}{lcl} d''' - b'' - a' & = & r - 2s + t \quad \text{By the 3d. Conc.} \\ d''' - b'' - a' + c'' & = & s - 2t + v \quad \text{By the 5th.} \end{array}$$

$$6^{\circ} \text{ Theref. } \underline{d''' - b'' - a' - c'' = r - 3s + 3t - v}$$

By the same process, if no Letter be particularly assigned to be in its place, the Probability that such of them as are assigned may be out of their places will likewise be found thus.

$$\begin{array}{lcl} -a' & = & 1 - r \quad \text{For } +a' \text{ and } -a' \text{ together make} \\ -a' + b'' & = & r - s \quad \text{[Unity.]} \end{array}$$

$$7^{\circ} \text{ Theref. } \underline{-a' - b'' = 1 - 2r + s}$$

$$\begin{array}{lcl} -a' - b'' & = & 1 - 2r + s \quad \text{By the 7th. Conc.} \\ -a' - b'' + c'' & = & r - 2s + t \quad \text{By the 3d. Conc.} \end{array}$$

$$8^{\circ} \text{ Theref. } \underline{-a' - b'' - c'' = 1 - 3r + 3s - t}$$

Now examining carefully all the foregoing Conclusions, it will be perceived, that when the Question runs barely upon the displacing any given number of Letters without requiring that any other should be in its place, but leaving it wholly indifferent, then the vulgar Algebraic Quantities which lie on the right hand of the Equations, begin constantly with Unity: It will also be perceived, that when one single Letter is assigned to be in its place, then those Quantities begin with r ; and that when two Letters are assigned to be in their places, they begin with s , and so on. Moreover 'tis obvious, that these Quantities change their signs alternately, and that the Numerical Coefficients which are prefixt to them are those of a Binomial Power, whose Index is equal to the number of Letters which are to be displaced.

PROBLEM XXVI.

ANY given number of different Letters a, b, c, d, e, f &c. being each of them repeated a certain number of times, and taken promiscuously as it Happens: To find the Probability that of some of those Sorts, some one Letter of each may be found in its proper place, and at the same time that of some other Sorts, no one Letter be found in its place.

SOLU-

SOLUTION.

Suppose n be the number of all the Letters, l the number of times that each Letter is repeated, and consequently $\frac{n}{l}$ the number of Sorts: Suppose also that p be the number of Sorts that are to have one Letter of each in its place; and q the number of Sorts of which no one Letter is to be found in its place. Let now the prescriptions given in the preceding Problem be followed in all respects, saving that r must here be made $= \frac{l}{n}$, $s = \frac{ll}{n \times n-1}$, $t = \frac{l^3}{n \times n-1 \times n-2}$ &c. and the Solution of any particular Case of the Problem will be obtained.

Thus if it were required to find the Probability that no Letter of any sort shall be in its place, the Probability thereof would be

$$1 - qr + \frac{q}{1} \times \frac{q-1}{2} s - \frac{q}{1} \times \frac{q-1}{2} \times \frac{q-2}{3} t \text{ \&c.}$$

But in this particular Case q would be equal to $\frac{n-l}{1}$, wherefore the foregoing Series might be changed into this, viz.

$$\frac{1}{2} \times \frac{n-l}{n-1} - \frac{1}{6} \times \frac{n-l \times n-2l}{n-1 \times n-2} + \frac{1}{24} \times \frac{n-l \times n-2l \times n-3l}{n-1 \times n-2 \times n-3} \text{ \&c.}$$

Corollary I. From hence it follows, that the Probability that one or more Letters indeterminately taken may be in their places will be

$$1 - \frac{1}{2} \times \frac{n-l}{n-1} + \frac{1}{6} \times \frac{n-l \times n-2l}{n-1 \times n-2} - \frac{1}{24} \times \frac{n-l \times n-2l \times n-3l}{n-1 \times n-2 \times n-3} \text{ \&c.}$$

Corollary II. The Probability that two or more Letters indeterminately taken may be in their places will be exprest as follows,

$$\frac{1}{2} \times \frac{n-l}{n-1} - \frac{2}{1 \times 3} \times \frac{n-2l}{n-2} A + \frac{3}{2 \times 4} \times \frac{n-3l}{n-3} B - \frac{4}{3 \times 5} \times \frac{n-4l}{n-4} C \\ + \frac{5}{4 \times 6} \times \frac{n-5l}{n-5} D \text{ \&c.}$$

Corollary III. The Probability that three or more Letters indeterminately taken may be in their places will be as follows,

$$\frac{1}{6} \times \frac{n-1 \times n-2}{n-1 \times n-2} - \frac{1}{1 \times 4} \times \frac{n-3}{n-3} A + \frac{1}{2 \times 5} \times \frac{n-4}{n-4} B \\ - \frac{1}{3 \times 6} \times \frac{n-5}{n-5} C + \frac{1}{4 \times 7} \times \frac{n-6}{n-6} D \&c.$$

Corollary IV. The Probability that four or more Letters, indeterminately taken, may be in their places will be thus exprest,

$$\frac{1}{24} \times \frac{n-1}{n-1} \times \frac{n-2}{n-2} \times \frac{n-3}{n-3} - \frac{1}{1 \times 5} \times \frac{n-4}{n-4} A + \frac{1}{2 \times 6} \times \frac{n-5}{n-5} B \\ - \frac{1}{3 \times 7} \times \frac{n-6}{n-7} C \&c.$$

The Law of the continuation of these Series being manifest, it will be easy to reduce them all to one general Series.

From what we have said it follows, that in a common Pack of 52 Cards, the Probability that one of the four Aces may be in the first place; one of the four Duces in the second; or one of the four Traes in the third; or that some one of any other sort may be in its place (making 13 different places in all) will be exprest by the Series exhibited in the first Corollary.

It follows likewise, that if there be two Packs of Cards, and that the Order of the Cards in one of the Packs be the Rule whereby to estimate the rank which the Cards of the same Suite and Name are to obtain in the other; the Probability that one Card or more, in one of the Packs, may be found in the same Position as the like Card in the other Pack, will be exprest by the Series belonging to the first Corollary, making $n = 52$ and $l = 1$: Which Series will in this Case be $1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720} \&c.$ whereof 52 Terms ought to be taken.

If the Terms of the foregoing Series are joined by couples, the Series will become,

$$\frac{1}{2} + \frac{1}{2 \times 4} + \frac{1}{2 \times 3 \times 4 \times 6} + \frac{1}{2 \times 3 \times 4 \times 5 \times 6 \times 8} + \frac{1}{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 10} \\ \&c. \text{ of which 26 Terms ought to be taken.}$$

But by reason of the great Convergency of the afore said Series, a few of its Terms will give a sufficient approximation

tion in all Cases required ; as appears by the following Operation,

$$\begin{array}{r}
 \frac{1}{2} = 0.500000 \\
 \frac{1}{2 \times 4} = 0.125000 \\
 \frac{1}{2 \times 3 \times 4 \times 6} = 0.006944 + \\
 \frac{1}{2 \times 3 \times 4 \times 5 \times 6 \times 8} = 0.000174 + \\
 \frac{1}{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 10} = 0.000002 + \\
 \hline
 \text{Sum} = 0.632129 +
 \end{array}$$

Wherefore the Probability that one or more like Cards in two different Packs may obtain the same Position, will be in all Cases very near 0.632; and the Odds that this will Happen once or oftner, as 632 to 368, or as 12 to 7 very near.

But the Odds that two or more like Cards in two different Packs will not obtain the same Position, are very nearly as 736 to 264 or 14 to 5.

Corollary V. If *A* and *B*, each holding a Pack of Cards, pull them out at the same time one after another, on condition that every time two like Cards are pulled out, *A* shall give *B* a Guinea; and it were required to find what consideration *B* ought to give *A* to Play on those terms: The Answer will be, One Guinea, let the number of Cards be what it will.

Corollary VI. If the number of Packs be given, the Probability that any given number of circumstances may Happen in them all, or in any of them, will be found easily by our method. Thus, if the number of the Packs be *k*, the Probability that one Card or more of the same Sute and Name, in every one of the Packs, may be in the same Position, will be exprest as follows.

$$\frac{1}{k-2} - \frac{1}{2 \times n \times n-1} k-2 + \frac{1}{6 \times n \times n-1 \times n-2} k-2 - \frac{1}{24 \times n \times n-1 \times n-2 \times n-3} k-2 \\
 \&c.$$

PRO.

PROBLEM XXVII.

IF *A* and *B* play together, each with a certain number of Bowls $= n$: What are their respective Probabilities of winning, supposing that each of them want a certain number of Games of being up?

SOLUTION.

First, the Probability that some Bowl of *B* may be nearer the Jack than any Bowl of *A* is $\frac{1}{2}$.

Secondly, Supposing one of his Bowls nearer the Jack than any Bowl of *A*, the number of his remaining Bowls is $n-1$, and the number of all the Bowls remaining between them is $2n-1$: Wherefore the Probability that some other of his Bowls may be nearer the Jack than any Bowl of *A* will be $\frac{n-1}{2n-1}$, from whence it follows, that the Probability of his winning two Bowls or more is $\frac{1}{2} \times \frac{n-1}{2n-1}$.

Thirdly, Supposing two of his Bowls nearer the Jack than any Bowl of *A*, the Probability that some other of his Bowls may be nearer the Jack than any Bowl of *A* will be $\frac{n-2}{2n-2}$; Wherefore the Probability of winning three Bowls or more is $\frac{1}{2} \times \frac{n-1}{2n-1} \times \frac{n-2}{2n-2}$: The continuation of which process is manifest.

Fourthly, The Probability that one single Bowl of *B* shall be nearer the Jack than any Bowl of *A* is $\frac{1}{2} - \frac{1}{2} \times \frac{n-1}{2n-1}$, or $\frac{1}{2} \times \frac{n}{2n-1}$; For, if from the Probability that one or more of his Bowls may be nearer the Jack than any Bowl of *A*, there be subtracted the Probability that two or more may be nearer, there remains the Probability of one single Bowl of *B* being nearer: In this Case *B* is said to Win one Bowl at an End.

Fifthly, The Probability that two Bowls of *B*, and not more, may be nearer the Jack than any Bowl of *A*, will be found to be $\frac{1}{2} \times \frac{n-1}{2n-1} \times \frac{n}{2n-2}$, in which Case *B* is said to win two Bowls at an End.

Sixthly,

Sixthly, The Probability that *B* may win three Bowls at an End will be found to be $\frac{1}{2} \times \frac{n-1}{2n-1} \times \frac{n-2}{2n-2} \times \frac{n}{2n-3}$. The process whereof is manifest.

The Reader may observe, that the foregoing Expressions might be reduced to fewer Terms; but leaving them unreduced, the Law of the process is thereby made more conspicuous.

Let it carefully be observ'd, when we mention henceforth the Probability of winning two Bowls, that the Sense of it ought to be extended to two Bowls or more; and that when we mention the winning two Bowls at an End, it ought to be taken in the common acceptation of two Bowls only: The like being to be observed in other Cases.

This Preparation being made; suppose, *First*, that *A* wants one Game of being up, and *B* two; and let it be required, in that circumstance, to determine their Probabilities of winning.

Let the whole Stake between them be supposed = 1. Then either *A* may win a Bowl, or *B* win one Bowl at an End, or *B* may win two Bowls.

In the first Case *B* loses his Expectation.

In the second Case he becomes intitled to $\frac{1}{2}$ of the Stake. But the Probability of this Case is $\frac{1}{2} \times \frac{n}{2n-1}$: wherefore his Expectation arising from that part of the Stake he will be intitled to, if this Case should Happen, and from the Probability of its Happening, will be $\frac{1}{4} \times \frac{n}{2n-1}$.

In the third Case *B* wins the whole Stake 1. But the Probability of this Case is $\frac{1}{2} \times \frac{n-1}{2n-1}$: wherefore the Expectation of *B* upon that account is $\frac{1}{2} \times \frac{n-1}{2n-1}$.

From this it follows that the whole Expectation of *B* is $\frac{1}{4} \times \frac{n}{2n-1} + \frac{1}{2} \times \frac{n-1}{2n-1}$ or $\frac{3}{4} \times \frac{n}{2n-1}$, or $\frac{3n-2}{8n-4}$; which

being subtracted from Unity, the remainder will be the Expectation of *A*, viz. $\frac{5n-2}{8n-4}$. It may therefore be concluded, that the Probabilities which *A* and *B* have of winning are respectively as $5n-2$ to $3n-2$.

'Tis remarkable, that the fewer the Bowls are, the greater is the proportion of the Odds; for if *A* and *B* play with single

single Bowls, the proportion will be as 3 to 1; if they play with two Bowls each, the proportion will be as 2 to 1; if with three Bowls each, the proportion will be as 13 to 7: yet let the number of Bowls be never so great, that proportion will not descend so low as 5 to 3.

Secondly, Suppose *A* wants one Game of being up, and *B* three; then either *A* may win a Bowl, or *B* win one Bowl at an End, or two Bowls at an End, or three Bowls.

In the first Case *B* loses his Expectation.

If the second Case Happens, then *B* will be in the circumstance of wanting but two to *A*'s one; in which Case his Expectation will be $\frac{3n-2}{8n-4}$, as it has been before determined: but the Probability that this Case may Happen is $\frac{1}{2} \times \frac{n}{2n-1}$; wherefore the Expectation of *B*, arising from the prospect of this Case, will be $\frac{1}{2} \times \frac{n}{2n-1} \times \frac{3n-2}{8n-4}$.

If the third Case Happen, then *B* will be intitled to one half of the Stake: but the Probability of its Happening is $\frac{1}{2} \times \frac{n-1}{2n-1} \times \frac{n}{2n-2}$; wherefore the Expectation of *B* arising from the Prospect of this Case is $\frac{1}{4} \times \frac{n-1}{2n-1} \times \frac{n}{2n-2}$, or $\frac{1}{8} \times \frac{n}{2n-1}$.

If the fourth Case Happen, then *B* wins the whole Stake 1: but the Probability of its Happening is $\frac{1}{2} \times \frac{n-1}{2n-1} \times \frac{n-2}{2n-2}$, or $\frac{1}{4} \times \frac{n-2}{2n-1}$; wherefore the Expectation of *B* arising from the prospect of this Case will be found to be $\frac{1}{4} \times \frac{n-2}{2n-1}$.

From this it follows, that the whole Expectation of *B* will be $\frac{9nn-13n+4}{8 \times 2n-1}$; which being subtracted from Unity, the remainder will be the Expectation of *A*, viz. $\frac{23nn-19n+4}{8 \times 2n-1}$.

It may therefore be concluded, that the Probabilities which *A* and *B* have of winning are respectively as $23nn-19n+4$ to $9nn-13n+4$.

N. B. If *A* and *B* play only with One Bowl each, the Expectation of *B* deduced from the foregoing Theorem would be found = 0. which we know from other principles ought to be = $\frac{1}{8}$. The reason of which is that the Case of winning Two Bowls at an End, and the Case of winning

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Three

Three Bowls at an End, enter this conclusion, which Cases do not belong to the supposition of playing with single Bowls: wherefore excluding those two Cases, the Expectation of *B* will be found to be $\frac{1}{2} \times \frac{n}{2n-1} \times \frac{3n-2}{8n-4} = \frac{1}{8}$, which will appear if *n* be made = 1. Yet the Expectation of *B*, in the Case of two Bowls, would be rightly determined, tho' the Case of winning Three Bowls at an End enters it: The reason of which is, that the Probability of winning Three Bowls at an End is =, $\frac{1}{4} \times \frac{n-2}{2n-1}$, which in the Case of Two Bowls becomes = 0, so that the general Expression is not thereby disturbed.

After what we have said, it will be easy to extend this way of Reasoning to any circumstance of Games wanting between *A* and *B*; by making the Solution of each simpler Case subservient to the Solution of that which is immediately more compound.

Having given formerly the Solution of this Problem, proposed to me by the Honourable *Frances Robarts*, in the *Philosophical Transactions* Number 339; I there said, by way of Corollary, that if the proportion of Skill in the Gamesters were given, the Problem might also be Solved; since which time *Mr de Monmort*, in the second Edition of a Book by him Published upon the subject of Chance, has thought it worth his while to Solve this Problem as it is extended to the consideration of the Skill, and to carry his Solution to a very great number of Cases, giving also a Method by which it may still be carried farther: I very willingly acknowledge his Solution to be extremely good, and own that he has in this, as well as in a great many other things, shewn himself entirely master of the doctrine of Combinations, which he has employed with very great Industry and Sagacity.

The Solution of this Problem, as it is restrained to an equality of Skill, was in my *Specimen* deduced from the Method of Combinations; but the Solution which is given of it in this place, is deduced from a Principle which has more of simplicity in it, being that by the help of which I have Demonstrated the Doctrine of Permutations and Combinations: Wherefore to make it as familiar as possible, and to shew its vast extent, I shall now apply it to the general Solution of

of this Problem, taking in the consideration of the Skill of the Gamesters.

But before I proceed I think it necessary to define what I call Skill: *viz.* That it is the proportion of Chances which the Gamesters may be supposed to have for winning a single Game with one Bowl each.

PROBLEM XXVIII.

IF *A* and *B*, whose proportion of Skill is as *a* to *b*, play together, each with a certain number of Bowls: What are their respective Probabilities of winning, supposing each of them to want a certain number of Games of being up?

SOLUTION.

First, The Chance of *B* for winning one single Bowl being *b*, and the number of his Bowls being *n*, it follows that the sum of all his Chances is nb ; and for the same reason the sum of all the Chances of *A* is na : wherefore the sum of all the Chances for winning one Bowl or more is $na + nb$; which for brevity sake we may call *s*. From whence it follows, that the Probability which *B* has of winning one Bowl or more is $\frac{nb}{s}$.

Secondly, Supposing one of his Bowls nearer the Jack than any of the Bowls of *A*, the number of his remaining Chances is $n - 1 \times b$; and the number of Chances remaining between them is $s - b$: wherefore the Probability that some other of his Bowls may be nearer the Jack than any Bowl of *A* will be $\frac{n - 1 \times b}{s - b}$: From whence it follows, that the Probability of his winning Two Bowls or more is $\frac{nb}{s} \times \frac{n - 1 \times b}{s - b}$.

Thirdly, Supposing Two of his Bowls nearer the Jack than any of the Bowls of *A*, the number of his remaining Chances is $n - 2 \times b$; and the number of Chances remaining between them is $s - 2b$: wherefore the Probability that some other of his Bowls may be nearer the Jack than any Bowl of *A* will be $\frac{n - 2 \times b}{s - 2b}$. From whence it follows, that the Probability

Probability of his winning Three Bowls or more is $\frac{nb}{s} \times \frac{n-1 \times b}{s-b} \times \frac{n-2 \times b}{s-2b}$; the continuation of which process is manifest.

Fourthly, If from the Probability which *B* has of winning One Bowl or more, there be subtracted the Probability which he has of winning Two or more, there will remain the Probability of his winning One Bowl at an End: Which therefore will be found to be $\frac{nb}{s} - \frac{nb}{s} \times \frac{n-1 \times b}{s-b}$ or $\frac{nb}{s} \times \frac{s-nb}{s-b}$ or $\frac{nb}{s} \times \frac{an}{s-b}$.

Fifthly, For the same reason as above, the Probability which *B* has of winning Two Bowls at an End will be found to be $\frac{nb}{s} \times \frac{n-1 \times b}{s-b} \times \frac{an}{s-2b}$.

Sixthly, And for the same reason likewise, the Probability which *B* has of winning Three Bowls at an End will be found to be $\frac{nb}{s} \times \frac{n-1 \times b}{s-b} \times \frac{n-2 \times b}{s-2b} \times \frac{an}{s-3b}$. The continuation of which process is manifest.

N. B. The same Expectations which denote the Probability of any circumstance of *B*, will denote likewise the Probability of the like circumstance of *A*, only changing *b* into *a* and *a* into *b*.

These Things being premised, Suppose *First*, that each of them wants one Game of being up; 'tis plain that the Expectations of *A* and *B* are respectively $\frac{an}{s}$ and $\frac{bn}{s}$. Let this Expectation of *B* be called *P*.

Secondly, Suppose *A* wants One Game of being up and *B* Two, and let the Expectation of *B* be required: Then either *A* may win a Bowl, or *B* win One Bowl at an End, or *B* win Two Bowls.

If the first Case Happens, *B* loses his Expectation.

If the second Happens, he gets the Expectation *P*; but the Probability of this Case is $\frac{nb}{s} \times \frac{an}{s-b}$: wherefore the Expectation of *B* arising from the possibility that it may so Happen is $\frac{nb}{s} \times \frac{an}{s-b} \times P$.

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If the third Case Happens, he gets the whole Stake 1; but the Probability of this Case is $\frac{nb}{s} \times \frac{n-b}{s-b}$, wherefore the Expectation of B arising from the Probability of this Case is $\frac{nb}{s} \times \frac{n-b}{s-b} \times 1$.

From which it follows that the whole Expectation of B will be $\frac{nb}{s} \times \frac{an}{s-b} P + \frac{nb}{s} \times \frac{n-b}{s-b}$. Let this Expectation be called Q.

Thirdly, Suppose A to want One Game of being up, and B Three. Then either B may win One Bowl at an End, in which Case he gets the Expectation Q; or Two Bowls at an End, in which Case he gets the Expectation P; or Three Bowls in which Case he gets the whole Stake 1. Wherefore the

Expectation of B will be found to be $\frac{nb}{s} \times \frac{an}{s-b} \times Q$
 $+ \frac{nb}{s} \times \frac{n-1 \times b}{s-b} \times \frac{an}{s-2b} \times P + \frac{nb}{s} \times \frac{n-1 \times b}{s-b} \times \frac{n-2 \times b}{s-2b}$.

An infinite number of these Theorems may be formed in the same manner, which may be continued by inspection, having well observed how each of them is deduced from the preceding.

If the number of Bowls were unequal, so that A had m Bowls and B n Bowls; Supposing $ma + nb = s$, other Theorems might be found to answer that inequality: And if that inequality should not be constant, but vary at pleasure; cther Theorems might also be formed to answer that Variation of inequality, by following the same way of arguing. And if Three or more Gamesters were to play together under any circumstance of Games wanting, and of any given proportion of Skill, their Probabilities of winning might be determined after the same manner.

PROBLEM XXIX.

TO find the Expectation of A when with a Die of any given number of Faces he undertakes to sling any determinate number of them in any given number of Casts.

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SOLUTION.

LET $p+1$ be the number of all the Faces in the Die, n the number of Casts, f the number of Faces which he undertakes to fling.

The number of Chances for an Ace to come up once or more in any number of Casts n , is $\overline{p+1}^n - p^n$: As has been proved in the Introduction.

Let the Duce, by thought, be expunged out of the Die, and thereby the number of its Faces reduced to p , then the number of Chances for the Ace to come up will at the same time be reduced to $p^n - \overline{p-1}^n$. Let now the Duce be restored, and the number of Chances for the Ace to come up without the Duce, will be the same as if the Duce were expunged. But if from the number of Chances for the Ace to come up with or without the Duce, *viz.* from $\overline{p+1}^n - p^n$ be subtracted the number of Chances for the Ace to come up without the Duce, *viz.* $p^n - \overline{p-1}^n$, there will remain the number of Chances for the Ace and Duce to come up once or more, which consequently will be $\overline{p+1}^n - 2 \times p^n + \overline{p-1}^n$.

By the same way of arguing it will be proved, that the number of Chances for the Ace and Duce to come up without the Trae will be $p^n - 2 \times \overline{p-1}^n + \overline{p-2}^n$, and consequently, that the number of Chances for the Ace, the Duce and Trae to come up once or more, will be the difference between $\overline{p+1}^n - 2 \times p^n + \overline{p-1}^n$ and $p^n - 2 \times \overline{p-1}^n + \overline{p-2}^n$; which therefore is $\overline{p+1}^n - 3 \times p^n + 3 \times \overline{p-1}^n - \overline{p-2}^n$.

Again it may be proved that the number of Chances for the Ace, the Duce, the Trae and Quater to come up, is $\overline{p+1}^n - 4 \times p^n + 6 \times \overline{p-1}^n - 4 \times \overline{p-2}^n + \overline{p-3}^n$; the continuation of which Process is manifest.

Wherefore if all the Powers $\overline{p+1}^n, p^n, \overline{p-1}^n, \overline{p-2}^n, \overline{p-3}^n$ &c. with the Signs alternately Positive and Negative, be written in Order, and to those Powers there be prefixt the respective Coefficients of a Binomial raised to the Power f ; the sum of all those Terms will be the Numerator of the Expectation of A , of which the Denominator will be $\overline{p+1}^n$.

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EXAMPLE I.

LET Six be the number of Faces in the Die, and let *A* undertake in Eight Casts to fling both an Ace and a Duce: Then his Expectation will be $\frac{6^8 - 2 \times 5^8 + 4^8}{6^8}$
 $= \frac{964502}{1680216} = \frac{4}{7}$ nearly.

EXAMPLE II.

IF *A* undertake with a common Die to fling all the Faces in 12 Casts, his Expectation will be found to be $\frac{6^{12} - 6 \times 5^{12} + 15 \times 4^{12} - 20 \times 3^{12} + 15 \times 2^{12} - 6 \times 1^{12}}{6^{12}} = \frac{10}{23}$ nearly.

EXAMPLE III.

IF *A* with a Die of 36 Faces undertake to fling two given Faces in 43 Casts; or, which is the same thing, if with two common Dice he undertake in 43 Casts to fling Two Aces at one time, and Two Sixes at another time, his Expectation will be $\frac{36^{43} - 2 \times 35^{43} + 34^{43}}{36^{43}} = \frac{49}{100}$ nearly.

N. B. The parts of which these Expectations are compounded, are easily obtained by the help of a Table of Logarithms.

PROBLEM XXX.

TO find in how many Trials it will be probable that *A* with a Die of any given number of Faces shall throw any proposed number of them.

SOLUTION.

LET $p+1$ be the number of Faces in the Die, and f the number of Faces which are to be thrown. Divide the Logarithm of $\frac{1}{1 - \sqrt[p]{\frac{1}{p+1}}}$ by the Logarithm of $\frac{p+1}{p}$, and the

the Quotient will express nearly the number of Trials requisite, to make it as probable that the proposed Faces may be thrown as not.

DEMONSTRATION.

Suppose Six to be the number of Faces which are to be thrown, and n the number of Trials: Then by what has been demonstrated in the preceding Problem, the Expectation of A will be,

$$\frac{p+1^n - 6 \times p^n + 15 \times p-1^n - 20 \times p-2^n + 15 \times p-3^n - 6 \times p-4^n + p-5^n}{p+1^n}$$

Let it be supposed that the Terms $p+1, p, p-1, p-2$ &c. are in Geometric Progression (which supposition will very little err from the truth, especially if the proportion of p to 1 be not very small). Let now r be written instead of $\frac{p+1}{p}$, and then the Expectation of A will be changed into $1 - \frac{6}{r^n} + \frac{15}{r^{2n}} - \frac{20}{r^{3n}} + \frac{15}{r^{4n}} - \frac{6}{r^{5n}} + \frac{1}{r^{6n}}$,

or $1 - \frac{1}{r^n}^6$. But this Expectation of A ought to be made equal to $\frac{1}{2}$, since by supposition he has an equal Chance to win or lose: Hence will arise the Equation $1 - \frac{1}{r^n}^6 = \frac{1}{2}$

or $r^n = \frac{1}{1 - \sqrt[6]{\frac{1}{2}}}$, from which it may be concluded that

$n \times \text{Log. } r$, or $n \times \text{Log. } \frac{p+1}{p} = \text{Log. } \frac{1}{1 - \sqrt[6]{\frac{1}{2}}}$, and consequent-

ly that n is equal to the Logarithm of $\frac{1}{1 - \sqrt[6]{\frac{1}{2}}}$ divided by the

Logarithm of $\frac{6}{5} = \frac{p+1}{p}$. And the same Demonstration will hold in any other Case.

EXAMPLE I.

TO find in how many Trials A may with equal Chance undertake to throw all the Faces of a common Die.

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The Logarithm of $\frac{1}{1-\sqrt{\frac{6}{5}}}$ = 0.9621753; the Logarithm of $\frac{p+1}{p}$ or $\frac{6}{5}$ = 0.0791812: Wherefore $n = \frac{0.9621753}{0.0791812} = 12+$. From hence it may be concluded that in 12 Casts *A* has the worst of the Lay, and in 13 the best of it.

EXAMPLE II.

TO find in how many Trials, *A* may with equal Chance, with a Die of Thirty-six Faces, undertake to throw Six determinate Faces; or, in how many Trials he may with a Pair of common Dice undertake to throw all the Doublets.

The Logarithm of $\frac{1}{1-\sqrt{\frac{6}{5}}}$ being 0.9621753, and the Logarithm of $\frac{p+1}{p}$ or $\frac{36}{35}$ being 0.0122345; it follows that the number of Casts requisite to that effect is $\frac{0.9621753}{0.0122345}$ or 79 nearly.

But if it were the Law of the Play, that the Doublets must be thrown in a given Order, and that any Doublet Happening to be thrown out of its turn should go for nothing; then the throwing of the Six Doublets would be like the throwing of the two Aces Six times; to produce which effect the number of Casts requisite would be found by Multiplying 35 by 5.668, as appears from our VIIth. Problem, and consequently would be about 198.

N. B. the Fraction $\frac{1}{1-\sqrt{\frac{f}{5}}}$ may be reduced to $\frac{\sqrt{f_2}}{\sqrt{f_2}-1}$ which will Facilitate the taking of its Logarithm.

PROBLEM XXXI.

IF *A, B, C* Play together on the following conditions; First, that they shall each of them Stake 1 l. Secondly, that *A* and *B* shall begin the Play; Thirdly, that the Loser shall yield his place to the third Man, which is to be observed constantly afterwards; Fourthly, that

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that the Loser shall be fined a certain Sum p , which is to serve to increase the common Stock; Lastly, that he shall Win the whole Sum deposited at first, and increased by the several Fines, who shall first beat the other two successively: 'Tis demanded what is the Advantage of A and B , whom we suppose to begin the Play.

SOLUTION.

Let BA signifie that B beats A , and AC that A beats C ; and let always the first Letter denote the Winner, and the second the Loser.

Let us suppose that B beats A the first time: Then let us inquire what the Probability is that the Set shall be ended in any given number of Games; and also what is the Probability which each Gamester has of winning the Set in that given number of Games.

First, If the Set be ended in two Games, B must necessarily be the winner; for by Hypothesis he wins the first time: Which may be expressed as follows.

$$\begin{array}{c|c} 1 & BA \\ 2 & \hline & BC \end{array}$$

Secondly, If the Set be ended in Three Games, C must be the winner; as appears by the following Scheme.

$$\begin{array}{c|c} 1 & BA \\ 2 & \hline 3 & CB \\ & CA \end{array}$$

Thirdly, If the Set be ended in Four Games, A must be the winner; as appears by this Scheme.

$$\begin{array}{c|c} 1 & BA \\ 2 & \hline 3 & CB \\ 4 & \hline & AC \\ & AB \end{array}$$

Fourthly, If the Set be ended in Five Games, B must be the winner; which is thus expressed,

$$1 \mid BA$$

1		<u>BA</u>
2		<u>CB</u>
3		AC
4		BA
5		BC

Fifthly, If the Set be ended in Six Games, C must be the winner; as will appear by still following the same Process, thus,

1		<u>BA</u>
2		<u>CB</u>
3		AC
4		BA
5		CB
6		CA

And this Process recurring continually in the same Order needs not be prosecuted any farther.

Now the Probability that the first Scheme shall take place is $\frac{1}{2}$, in consequence of the supposition that B beats A the first time; it being an equal Chance whether B beat C, or C beat A.

And the Probability that the second Scheme shall take place is $\frac{1}{4}$: For the Probability of C beating B is $\frac{1}{2}$, and that being supposed, the Probability of his beating A will also be $\frac{1}{2}$; wherefore the Probability of B beating C, and then A, will be $\frac{1}{2} \times \frac{1}{2}$ or $\frac{1}{4}$.

And from the same considerations the Probability that the Third Scheme shall take place is $\frac{1}{8}$: and so on.

Hence it will be easie to compose a Table of the Probabilities which B, C, A have of winning the Set in any given number of Games; and also of their Expectations: Which Expectations are the Probabilities of winning Multiplied by the Stock Three deposited at first, and increased successively by the several Fines.

TABLE

TABLE of the Probabilities, &c.

	B	C	A
2	$\frac{1}{2} \times 3+2p$	• • • • •	• • • • •
3	• • • • •	$\frac{1}{4} \times 3+3p$	• • • • •
4	• • • • •	• • • • •	$\frac{1}{8} \times 3+4p$
5	$\frac{1}{16} \times 3+5p$	• • • • •	• • • • •
6	• • • • •	$\frac{1}{32} \times 3+6p$	• • • • •
7	• • • • •	• • • • •	$\frac{1}{64} \times 3+7p$
8	$\frac{1}{128} \times 3+8p$	• • • • •	• • • • •
9	• • • • •	$\frac{1}{256} \times 3+9p$	• • • • •
10	• • • • •	• • • • •	$\frac{1}{512} \times 3+10p$
11	$\frac{1}{1024} \times 3+11p$	• • • • •	• • • • •
&c.			

Now the several Expectations of *B*, *C*, *A* may be summed up by the following Lemma.

L E M M A.

$\frac{n}{b} + \frac{n+d}{bb} + \frac{n+2d}{b^3} + \frac{n+3d}{b^4} + \frac{n+4d}{b^5} \&c. \text{ Ad infinitum}$
 is equal to $\frac{n}{b-1} + \frac{d}{(b-1)^2}$.

Let the Expectations of *B* be divided into two Series, viz.

$$\begin{aligned} & \frac{3}{2} + \frac{3}{16} + \frac{3}{128} + \frac{3}{1024} \&c. \\ & + \frac{2p}{2} + \frac{5p}{16} + \frac{8p}{128} + \frac{11p}{1024} \&c. \end{aligned}$$

The first Series constitutes a Geometric Progression continually decreasing, whose sum will be found to be $\frac{12}{7}$.

The second Series may be reduced to the form of the Series in our Lemma, and may be thus exprest,

$$\frac{p}{2} \times$$

$\frac{p}{2} \times \frac{2}{1} + \frac{5}{8} + \frac{8}{8^2} + \frac{11}{8^3} + \frac{14}{8^4}$ &c. Wherefore dividing the whole by $\frac{p}{2}$, and laying aside the Term 2, we shall have the Series $\frac{5}{8} + \frac{8}{8^2} + \frac{11}{8^3} + \frac{14}{8^4}$, &c. which has the same form as the Series of the Lemma, and may be compared with it: Let therefore n be made $= 5$, $d = 3$ and $b = 8$, and the sum of this Series will be $\frac{5}{7} + \frac{3}{49}$, or $\frac{38}{49}$; to this adding the first Term 2, which had been laid aside, the new sum will be $\frac{136}{49}$; and that being Multiplied by $\frac{p}{2}$, the Product will be $\frac{68}{49}p$, which is the sum of the second Series expressing the Expectations of B : From hence it may be concluded, that all the Expectations of B contained in both the abovementioned Series will be equal to $\frac{12}{7} + \frac{68}{49}p$.

And by the help of the foregoing Lemma it will be found likewise that all the Expectations of C will be equal to $\frac{6}{7} + \frac{48}{49}p$.

It will also be found that all the Expectations of A will be $= \frac{3}{7} + \frac{31}{49}p$.

Hitherto we have determined the several Expectations of the Gamesters, upon the sum by them deposited at first, as also upon the Fines by which the common Stock is increased: It remains now to Estimate the several Risks of their being Fined; that is to say, the sum of the Probabilities of their being Fined multiplied by the respective Quantities of the Fine.

Now after the supposition made of A being beat the first time, by which he is obliged to lay down his Fine p , B and C have an equal Chance of being Fined after the second Game, which makes the Risk of each to be $= \frac{1}{2}p$, as appears by the following Scheme.

$$\frac{BA}{CB} \text{ or } \frac{BA}{BC}$$

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In the like manner, it will be found that both *C* and *A* have one Chance in four for their being Fined after the Third Game, and consequently that the Risk of each is $\frac{1}{4}p$, according to the following Scheme.

$$\frac{BA}{CB} \text{ or } \frac{BA}{CA}$$

And by the like Process it will be found that the Risk of *B* and *C* after the fourth Game is $\frac{1}{8}p$.

Hence it will be easie to compose the following Table which expresses the Risks of each Gamester.

TABLE of RISKS.

	B	C	A
2	$\frac{1}{2}p$	$\frac{1}{2}p$	-----
3	-----	$\frac{1}{4}p$	$\frac{1}{4}p$
4	$\frac{1}{8}p$	-----	$\frac{1}{8}p$
5	$\frac{1}{16}p$	$\frac{1}{16}p$	-----
6	-----	$\frac{1}{32}p$	$\frac{1}{32}p$
7	$\frac{1}{64}p$	-----	$\frac{1}{64}p$
8	$\frac{1}{128}p$	$\frac{1}{128}p$	-----
9	-----	$\frac{1}{256}p$	$\frac{1}{256}p$
Gr.			

In the Column belonging to *B*, if the vacant places were filled up, and the Terms $\frac{1}{4}p$, $\frac{1}{32}p$, $\frac{1}{256}p$ &c. were Interpoled, the Sum of the Risks of *B* would compose one uninterrupted Geometric Progression, whose Sum would be $= p$; But the Terms interpoled constitute a Geometric Progression whose Sum is $= \frac{2}{7}p$: Wherefore, if from p there be subtracted $\frac{2}{7}p$, there will remain $\frac{5}{7}p$ for the Sum of the Risks of *B*.

In like manner it will be found that the Sum of the Risks of *C* will be $= \frac{6}{7}p$. And

And the Sum of the Risks of *A*, after his being Fined the first time, will be $= \frac{3}{7}p$.

Now if from the several Expectations of the Gamesters there be subtracted each Man's Stake, as also the Sum of his Risks, there will remain the clear Gain or Loss of each of them.

$$\begin{array}{lcl} \text{Wherefore, from the Expectations of } B & = & \frac{12}{7} + \frac{68}{49}p \\ \text{Subtracting first his Stake} & = & 1 \\ \text{Then the Sum of his Risks} & = & \frac{5}{7}p \end{array}$$

$$\text{There remains the clear Gain of } B = \frac{5}{7} + \frac{33}{49}p$$

$$\begin{array}{lcl} \text{Likewise, from the Expectations of } C & = & \frac{6}{7} + \frac{48}{49}p \\ \text{Subtracting first his Stake} & = & 1 \\ \text{Then the Sum of his Risks} & = & \frac{6}{7}p \end{array}$$

$$\text{There remains the clear Gain of } C = -\frac{1}{7} + \frac{6}{49}p$$

$$\begin{array}{lcl} \text{In like manner, from the Expectation of } A & = & \frac{3}{7} + \frac{31}{49}p \\ \text{Subtracting, First, his Stake} & = & 1 \\ \text{Secondly, the Sum of his Risks} & = & \frac{3}{7}p \\ \text{Lastly, the Fine } p \text{ due to the } \} & = & p \\ \text{Stock by the Loss of the first Game} \} & & \end{array}$$

$$\text{There remains the clear Gain of } A = -\frac{4}{7} - \frac{39}{49}p$$

But we have supposed in the beginning of the Game that *A* was beat; whereas *A* had the same Chance to beat *B*, as *B* had to beat him: Wherefore dividing the Sum of the Gains of *B* and *A* into two equal Parts, each part will be $\frac{1}{14} - \frac{3}{49}p$; which consequently must be reputed to be as the Gain of each of them.

Corollary I. The Gain of *C* being $-\frac{1}{7} + \frac{6}{49}p$, Let that be made $= 0$. Then *p* will be found $= \frac{7}{6}$. If therefore the Fine has the same proportion to each Man's Stake as 7 has to 6, the Gamesters play all upon equal Terms: But if the Fine bears a less proportion to the Stake than 7

to.

In the like manner, it will be found that both *C* and *A* have one Chance in four for their being Fined after the Third Game, and consequently that the Risk of each is $\frac{1}{4}p$, according to the following Scheme.

$$\frac{BA}{CB} \text{ or } \frac{BA}{CB} \\ AC \quad CA$$

And by the like Process it will be found that the Risk of *B* and *C* after the fourth Game is $\frac{1}{8}p$.

Hence it will be easie to compose the following Table which expresses the Risks of each Gamester.

TABLE of RISKS.

	B	C	A
2	$\frac{1}{2}p$	$\frac{1}{2}p$	-----
3	-----	$\frac{1}{4}p$	$\frac{1}{4}p$
4	$\frac{1}{8}p$	-----	$\frac{1}{8}p$
5	$\frac{1}{16}p$	$\frac{1}{16}p$	-----
6	-----	$\frac{1}{32}p$	$\frac{1}{32}p$
7	$\frac{1}{64}p$	-----	$\frac{1}{64}p$
8	$\frac{1}{128}p$	$\frac{1}{128}p$	-----
9	-----	$\frac{1}{256}p$	$\frac{1}{256}p$
&c.			

In the Column belonging to *B*, if the vacant places were filled up, and the Terms $\frac{1}{4}p$, $\frac{1}{32}p$, $\frac{1}{256}p$ &c. were Interpoled, the Sum of the Risks of *B* would compose one uninterrupted Geometric Progression, whose Sum would be $= p$; But the Terms interpoled constitute a Geometric Progression whose Sum is $= \frac{2}{7}p$: Wherefore, if from p there be subtracted $\frac{2}{7}p$, there will remain $\frac{5}{7}p$ for the Sum of the Risks of *B*.

In like manner it will be found that the Sum of the Risks of *C* will be $= \frac{6}{7}p$. And

And the Sum of the Risks of *A*, after his being Fined the first time, will be $= \frac{3}{7} p$.

Now if from the several Expectations of the Gamesters there be subtracted each Man's Stake, as also the Sum of his Risks, there will remain the clear Gain or Loss of each of them.

Wherefore, from the Expectations of <i>B</i>	$= \frac{12}{7} + \frac{63}{49} p$
Subtracting <i>first</i> his Stake	$= 1$
Then the Sum of his Risks	$= \frac{3}{7} p$

There remains the clear Gain of *B* $= \frac{5}{7} + \frac{33}{49} p$

Likewise, from the Expectations of <i>C</i>	$= \frac{6}{7} + \frac{43}{49} p$
Subtracting <i>first</i> his Stake	$= 1$
Then the Sum of his Risks	$= \frac{6}{7} p$

There remains the clear Gain of *C* $= -\frac{1}{7} + \frac{6}{49} p$

In like manner, from the Expectation of <i>A</i>	$= \frac{3}{7} + \frac{31}{49} p$
Subtracting, <i>First</i> , his Stake	$= 1$
<i>Secondly</i> , the Sum of his Risks	$= \frac{3}{7} p$
<i>Lastly</i> , the Fine <i>p</i> due to the } Stock by the Loss of the first Game }	$= p$

There remains the clear Gain of *A* $= -\frac{4}{7} - \frac{39}{49} p$

But we have supposed in the beginning of the Game that *A* was beat; whereas *A* had the same Chance to beat *B*, as *B* had to beat him: Wherefore dividing the Sum of the Gains of *B* and *A* into two equal Parts, each part will be $\frac{1}{14} - \frac{3}{49} p$; which consequently must be reputed to be as the Gain of each of them.

Corollary I. The Gain of *C* being $-\frac{1}{7} + \frac{6}{49} p$, Let that be made $= 0$. Then *p* will be found $= \frac{7}{6}$. If therefore the Fine has the same proportion to each Man's Stake as 7 has to 6, the Gamesters play all upon equal Terms: But if the Fine bears a less proportion to the Stake than 7

to 6, C has the disadvantage: Thus, Supposing $p = 1$, his Loss would be $\frac{1}{49}$. But if the Fine bears a greater proportion to the Stake than 7 to 6, C has the Advantage.

Corollary II. If the Stake were constant, that is, if there were no Fines, then the Probabilities of winning would be respectively proportional to the Expectations; wherefore supposing $p = 0$, the Expectations of the Gamesters, or their Probabilities of winning, will be as $\frac{12}{7}$, $\frac{6}{7}$, $\frac{3}{7}$, or, as 4, 2, 1: But the increase of the Stock causes no alteration in the Probabilities of winning, and consequently those Probabilities are, in the Case of this Problem, as 4, 2, 1; whereof the first belongs to B after his beating A the first time; the second to C , and the third to A : Wherefore 'tis Five to Two, before the Play begins, that either A or B wins the Set; and Five to Four that one of them, that shall be fixt upon, wins it.

Corollary III. If the proportion of Skill between the Gamesters A, B, C be as a, b, c respectively, and that the respective Probabilities of winning, in any number of Games after the first, wherein B is Supposed to beat A , be denoted by $B', B'', B''', B^{IV} \&c.$, $C'', C''', C^{IV}, C^{V} \&c.$, $A''', A^{IV}, A^{V}, A^{VI} \&c.$ it will be found, by the bare inspection of the Schemes belonging to the Solution of the foregoing Problem, that

$$B' = \frac{b}{b+c}$$

$$C'' = \frac{c}{c+b} \times \frac{c}{c+a}$$

$$A''' = \frac{c}{c+b} \times \frac{a}{a+c} \times \frac{a}{a+b}$$

$$B^{IV} = \frac{c}{c+b} \times \frac{a}{a+c} \times \frac{b}{b+a} \times \frac{b}{b+c}$$

$$C^V = \frac{c}{c+b} \times \frac{a}{a+c} \times \frac{b}{b+a} \times \frac{c}{c+b} \times \frac{c}{c+a}$$

$$A^{VI} = \frac{c}{c+b} \times \frac{a}{a+c} \times \frac{b}{b+a} \times \frac{c}{c+b} \times \frac{a}{a+c} \times \frac{a}{a+b}$$

&c.

Let $\frac{b}{b+a} \times \frac{c}{c+b} \times \frac{a}{a+c}$ be made $= m$; then it will plainly appear that the several Probabilities of winning will compose each of them a Geometric Progression, for

$$B''' =$$

$$\begin{array}{l} B''' = m B' \\ B'' = m B''' \\ B^x = m B'' \\ \&c. \end{array} \left| \begin{array}{l} C^v = m C'' \\ C''' = m C^v \\ C^x = m C''' \\ \&c. \end{array} \right| \begin{array}{l} A'' = m A''' \\ A'^x = m A'' \\ A^{x'} = m A'^x \\ \&c. \end{array}$$

Hence a Table of Expectations and Risks may easily be formed as above; and the rest of the Solution carried on by following exactly the steps of the former.

When the Solution is brought to its conclusion, it will be necessary to make an allowance for the supposition made that *B* beats *A* the first time, which may be done thus,

Let *P* be the Gain of *B*, when exprest by the Quantities *a*, *b*, *c*, and *Q* the Gain of *A*, when exprest by the same: Change *a* into *b* and *b* into *a*, in the Quantity *Q*; then the Quantity resulting from this Change will be the Gain of *B*, in case he be supposed to lose the first Game. Let this Quantity therefore be called *R*, and then the Gain of *B*, to be estimated before the Play begins, will be $\frac{bP + aR}{b + a}$.

PROBLEM XXXII.

IF Four Gamesters *A*, *B*, *C*, *D* Play on the conditions of the foregoing Problem, and he be to be reputed the Winner, who shall beat the other Three successively: What is the Advantage of *A* and *B*, whom we Suppose to begin the Play?

SOLUTION.

LET *BA* denote, as in the preceding Problem, that *B* beats *A*, and *AC* that *A* beats *C*; and generally let the first Letter always denote the Winner and the second the Loser.

Let it be Supposed also that *B* beats *A* the first time: Then let it be inquired what is the Probability that the Play shall be ended in any given number of Games; as also what is the Probability which each Gamester has of winning the Set in that given number of Games.

Z

First,

First, If the Set be ended in Three Games, *B* must necessarily be the winner: Since by Hypothesis he beats *A* the first Game, which is expressed as follows,

1		$\frac{BA}{}$
2		$\frac{BC}{}$
3		$\frac{BD}{}$

Secondly, If the Set be ended in Four Games, *C* must be the winner; as it thus appears.

1		$\frac{BA}{}$
2		$\frac{CB}{}$
3		$\frac{CD}{}$
4		$\frac{CA}{}$

Thirdly, If the Set be ended in Five Games, *D* will be the winner; for which he has two Chances, as it appears by the following Scheme.

1		$\frac{BA}{}$		$\frac{BA}{}$
2		$\frac{CB}{}$		$\frac{BC}{}$
3		$\frac{DC}{}$	or	$\frac{DB}{}$
4		$\frac{DA}{}$		$\frac{DA}{}$
5		$\frac{DB}{}$		$\frac{DC}{}$

Fourthly, If the Set be ended in Six Games, *A* will be the winner; and he has three Chances for it, which are thus collected,

1		$\frac{BA}{}$	$\frac{BA}{}$	$\frac{BA}{}$
2		$\frac{CB}{}$	$\frac{CB}{}$	$\frac{BC}{}$
3		$\frac{DC}{}$	$\frac{CD}{}$	$\frac{DB}{}$
4		$\frac{AD}{}$	$\frac{AC}{}$	$\frac{AD}{}$
5		$\frac{AB}{}$	$\frac{AB}{}$	$\frac{AC}{}$
6		$\frac{AC}{}$	$\frac{AD}{}$	$\frac{AB}{}$

Fifthly, If the Set be ended in Seven Games, then *B* will have three Chances to be the winner, and *C* will have two; thus,

1 | $\frac{BA}{}$

1	<u>BA</u>	<u>BA</u>	<u>BA</u>	<u>BA</u>	<u>BA</u>
2	<u>CB</u>	<u>CB</u>	<u>CB</u>	<u>BC</u>	<u>BC</u>
3	<u>DC</u>	<u>DC</u>	<u>CD</u>	<u>DB</u>	<u>DB</u>
4	<u>AD</u>	<u>DA</u>	<u>AC</u>	<u>AD</u>	<u>DA</u>
5	<u>BA</u>	<u>BD</u>	<u>BA</u>	<u>CA</u>	<u>CD</u>
6	<u>BC</u>	<u>BC</u>	<u>BD</u>	<u>CB</u>	<u>CB</u>
7	<u>BD</u>	<u>BA</u>	<u>BC</u>	<u>CD</u>	<u>CA</u>

Sixthly, If the Set be ended in Eight Games, then *D* will have two Chances to be the Winner, *C* will have three, and *B* also three, thus

1	<u>BA</u>	<u>BA</u>	<u>BA</u>	<u>BA</u>	<u>BA</u>	<u>BA</u>	<u>BA</u>	<u>BA</u>
2	<u>CB</u>	<u>CB</u>	<u>CB</u>	<u>CB</u>	<u>CB</u>	<u>BC</u>	<u>BC</u>	<u>BC</u>
3	<u>DC</u>	<u>DC</u>	<u>DC</u>	<u>CD</u>	<u>CD</u>	<u>DB</u>	<u>DB</u>	<u>DB</u>
4	<u>AD</u>	<u>AD</u>	<u>DA</u>	<u>AC</u>	<u>AC</u>	<u>AD</u>	<u>AD</u>	<u>DA</u>
5	<u>BA</u>	<u>AB</u>	<u>BD</u>	<u>BA</u>	<u>AB</u>	<u>CA</u>	<u>AC</u>	<u>CD</u>
6	<u>CB</u>	<u>CA</u>	<u>CB</u>	<u>DB</u>	<u>DA</u>	<u>BC</u>	<u>BA</u>	<u>BC</u>
7	<u>CD</u>	<u>CD</u>	<u>CA</u>	<u>DC</u>	<u>DC</u>	<u>BD</u>	<u>BD</u>	<u>BA</u>
8	<u>CA</u>	<u>CB</u>	<u>CD</u>	<u>DA</u>	<u>DB</u>	<u>BA</u>	<u>BC</u>	<u>BD</u>

Let now the Letters by which the winners are denoted be written in Order, prefixing to them the Numbers which express their several Chances for winning; in this manner,

3	1 B
4	1 C
5	2 D
6	3 A
7	3 B + 2 C
8	3 C + 2 D + 3 B
9	3 D + 2 A + 3 C + 3 D + 2 A
10	3 A + 2 B + 3 D + 3 A + 2 B + 3 A + 2 C + 3 D
&c.	

Then

Then Examining the formation of these Letters, it will appear; *First*, that the Letter *B* is always found so many times in any Rank, as the Letter *A* is found in the two preceding Ranks: *Secondly*, that *C* is found so many times in any Rank, as *B* is found in the preceding Rank, and *D* in the Rank before that. *Thirdly*, that *D* is found so many times in each Rank, as *C* is found in the preceding, and *B* in the Rank before that: And *Fourthly*, that *A* is found so many times in each, as *D* is found in the preceding Rank, and *C* in the Rank before that.

From whence it may be concluded, that the Probability which the Gamester *B* has of winning the Set, in any given number of Games, is $\frac{1}{2}$ of the Probability which *A* has of winning it one Game sooner, together with $\frac{1}{4}$ of the Probability which *A* has of winning it two Games sooner.

The Probability which *C* has of winning the Set, in any given number of Games, is $\frac{1}{2}$ of the Probability which *B* has of winning it one Game sooner, together with $\frac{1}{4}$ of the Probability which *D* has of winning it two Games sooner.

The Probability which *D* has of winning the Set, in any given number of Games, is $\frac{1}{2}$ of the Probability which *C* has of winning it one Game sooner, and also $\frac{1}{4}$ of the Probability which *B* has of winning it two Games sooner.

The Probability which *A* has of winning the Set, in any given number of Games, is $\frac{1}{2}$ of the Probability which *D* has of winning it one Game sooner, and also $\frac{1}{4}$ of the Probability which *C* has of winning it two Games sooner.

These things being observed, it will be easie to compose a Table of the Probabilities which *B*, *C*, *D*, *A* have of winning the Set in any given number of Games; as also of their Expectations, which will be as follows.

A T A.

TABLE of the Probabilities, &c.

		B	C	D	A
'	3	$\frac{1}{4} \times 4 + 3p$	-----	-----	-----
"	4	-----	$\frac{1}{8} \times 4 + 4p$	-----	-----
'''	5	-----	-----	$\frac{2}{16} \times 4 + 5p$	-----
''''	6	-----	-----	-----	$\frac{3}{32} \times 4 + 6p$
v	7	$\frac{3}{64} \times 4 + 7p$	$\frac{2}{64} \times 4 + 7p$	-----	-----
v'	8	$\frac{3}{128} \times 4 + 8p$	$\frac{3}{128} \times 4 + 8p$	$\frac{2}{128} \times 4 + 8p$	-----
v''	9	-----	$\frac{3}{256} \times 4 + 9p$	$\frac{6}{256} \times 4 + 9p$	$\frac{4}{256} \times 4 + 9p$
v'''	10	$\frac{4}{512} \times 4 + 10p$	$\frac{512}{10} \times 4 + 10p$	$\frac{6}{512} \times 4 + 10p$	$\frac{9}{512} \times 4 + 10p$
'x	11	$\frac{13}{1024} \times 4 + 11p$	$\frac{1024}{10} \times 4 + 11p$	$\frac{2}{1024} \times 4 + 11p$	$\frac{9}{1024} \times 4 + 11p$
x	12	$\frac{18}{2048} \times 4 + 12p$	$\frac{19}{2048} \times 4 + 12p$	$\frac{14}{2048} \times 4 + 12p$	$\frac{4}{2048} \times 4 + 12p$
&c.	&c.				

The Terms whereof each Column of this Table is composed, being not easily summable by any of the known Methods, it will be convenient, in order to find their Sums, to use the following *Analysis*.

Let $B' + B'' + B''' + B'''' + B^v + B^v$ &c. represent the respective Probabilities which *B* has of winning the Set, in any number of Games, answering to 3, 4, 5, 6, 7, 8 &c. and let the sum of these Probabilities *Ad infinitum* be supposed = y .

In the same manner, let $C' + C'' + C''' + C'''' + C^v + C^v$ &c. represent the Probabilities which *C* has of winning, which suppose = z .

Let the like Probabilities which *D* has of winning be represented by $D' + D'' + D''' + D'''' + D^v + D^v$ &c. which suppose = v .

Lastly, Let the Probabilities which *A* has of winning be represented by $A' + A'' + A''' + A'''' + A^v + A^v$ &c. which suppose = x .

Now from the Observations set down before the Table of Probabilities, it will follow, that

A a

B =

$$B' = B'$$

$$B'' = B''$$

$$B''' = \frac{1}{2} A'' + \frac{1}{4} A'$$

$$B'''' = \frac{1}{2} A''' + \frac{1}{4} A''$$

$$B^v = \frac{1}{2} A'''' + \frac{1}{4} A'''$$

$$B^{vi} = \frac{1}{2} A^v + \frac{1}{4} A''''$$

&c.

From which Scheme we may deduce the Equation following, $y = \frac{1}{4} + \frac{3}{4} x$: For the Sum of the Terms in the first Column is equal to the Sum of the Terms in the other two. But the Sum of the Terms in the first Column is y by Hypothesis; wherefore y ought to be made equal to the Sum of the Terms in the other two Columns.

In order to find the Sum of the Terms of the second Column, I argue thus,

$$A' + A'' + A''' + A'''' + A^v + A^{vi} \text{ is } = x \text{ by Hypoth.}$$

$$\text{or } A'' + A''' + A'''' + A^v + A^{vi} \text{ is } = x - A'$$

$$\text{and } \frac{1}{2} A'' + \frac{1}{2} A''' + \frac{1}{2} A'''' + \frac{1}{2} A^v + \frac{1}{2} A^{vi} \text{ is } = \frac{1}{2} x - \frac{1}{2} A'$$

Then adding $B' + B''$ on both sides of the last Equation, we shall have

$$B' + B'' + \frac{1}{2} A'' + \frac{1}{2} A''' + \frac{1}{2} A'''' + \frac{1}{2} A^v + \frac{1}{2} A^{vi} \text{ \&c.} \\ = \frac{1}{2} x - \frac{1}{2} A' + B' + B''.$$

But $A' = 0$, $B' = \frac{1}{4}$, $B'' = 0$, as appears from the Table: Wherefore the Sum of the Terms of the second Column is equal to $\frac{1}{2} x + \frac{1}{4}$.

The Sum of the Terms of the third Column is $\frac{1}{4} x$ by Hypothesis; and consequently the Sum of the Terms in the second and third Columns is $= \frac{3}{4} x + \frac{1}{4}$. From whence it follows that the Equation $y = \frac{1}{4} + \frac{3}{4} x$ had been rightly determined.

In

In the same manner, if we write

$$\begin{aligned} C' &= C' \\ C'' &= C'' \\ C''' &= \frac{1}{2} B'' + \frac{1}{4} D' \\ C'''' &= \frac{1}{2} B''' + \frac{1}{4} D'' \\ C^v &= \frac{1}{2} B'''' + \frac{1}{4} D''' \\ C^v &= \frac{1}{2} B^v + \frac{1}{4} D'''' \\ &\&c. \end{aligned}$$

By a reasoning like the former we shall at length come at the Equation $z = \frac{1}{2} y + \frac{1}{4} v$.

So likewise if we write

$$\begin{aligned} D' &= D' \\ D'' &= D'' \\ D''' &= \frac{1}{2} C'' + \frac{1}{4} B' \\ D'''' &= \frac{1}{2} C''' + \frac{1}{4} B'' \\ D^v &= \frac{1}{2} C'''' + \frac{1}{4} B''' \\ D^v &= \frac{1}{2} C^v + \frac{1}{4} B'''' \\ &\&c. \end{aligned}$$

We shall deduce the Equation $v = \frac{1}{2} z + \frac{1}{4} y$.

Lastly, if after the same manner we write

$$\begin{aligned} A' &= A' \\ A'' &= A'' \\ A''' &= \frac{1}{2} D'' + \frac{1}{4} C' \\ A'''' &= \frac{1}{2} D''' + \frac{1}{4} C'' \\ A^v &= \frac{1}{2} D'''' + \frac{1}{4} C''' \\ A^v &= \frac{1}{2} D^v + \frac{1}{4} C'''' \\ &\&c. \end{aligned}$$

We shall obtain the Equation $x = \frac{1}{2} v + \frac{1}{4} z$.

Now

Now these Four Equations being resolved, it will be found that

$$\begin{aligned} B' + B'' + B''' + B'''' + B^v + B^vi \text{ \&c.} &= y = \frac{56}{149}, \\ C' + C'' + C''' + C'''' + C^v + C^vi \text{ \&c.} &= z = \frac{36}{149}, \\ D' + D'' + D''' + D'''' + D^v + D^vi \text{ \&c.} &= v = \frac{32}{149}, \\ A' + A'' + A''' + A'''' + A^v + A^vi \text{ \&c.} &= x = \frac{25}{149}. \end{aligned}$$

These Values being once found, let b, c, d, a , which are commonly employed to denote known Quantities, be respectively substituted in the room of them; to the end that the Letters y, z, v, x may now be employed to denote other unknown Quantities.

Hitherto we have been determining the Probabilities of winning: But in order to find the Expectations of the Gamesters, each Term of the Series expressing these Probabilities, is to be multiplied by the respective Terms of the following Series; $4+3p, 4+4p, 4+5p, 4+6p, \text{ \&c.}$

The first part of each Product being no more than a Multiplication by 4, the sums of all the first parts of those Products are only the sums of the Probabilities multiplied by 4; and consequently are $4b, 4c, 4d$, and $4a$ respectively.

But to find the Sums of the other parts,

$$\begin{aligned} \text{Let } 3 B'p + 4 B''p + 5 B'''p + 6 B''''p \text{ \&c. be} &= p y, \\ 3 C'p + 4 C''p + 5 C'''p + 6 C''''p \text{ \&c.} &= p z, \\ 3 D'p + 4 D''p + 5 D'''p + 6 D''''p \text{ \&c.} &= p v, \\ 3 A'p + 4 A''p + 5 A'''p + 6 A''''p \text{ \&c.} &= p x, \end{aligned}$$

$$\text{Now Since } 3 B' = 3 B'$$

$$4 B'' = 4 B''$$

$$5 B''' = \frac{5}{2} A'' + \frac{5}{4} A'$$

$$6 B'''' = \frac{6}{2} A''' + \frac{6}{4} A''$$

$$7 B^v = \frac{7}{2} A'''' + \frac{7}{4} A'''$$

$$8 B^vi = \frac{8}{2} A^v + \frac{8}{4} A''''$$

\&c.

It

It follows, that $y = \frac{3}{4} + \frac{3}{4}x + a$. For the first Column is $= y$, by *Hypothesis*.

Again, $3 A' + 4 A'' + 5 A''' + 6 A'''' + 7 A^v \&c. = x$ by *Hypothesis*.

But $A' + A'' + A''' + A'''' + A^v \&c.$ has been found $= a$,

Wherefore adding these two Equations together, we shall have $4 A' + 5 A'' + 6 A''' + 7 A'''' + 8 A^v \&c. = x + a$.
or $\frac{4}{2} A' + \frac{5}{2} A'' + \frac{6}{2} A''' + \frac{7}{2} A'''' + \frac{8}{2} A^v \&c. = \frac{1}{2}x + \frac{1}{2}a$.

Now the Terms of this last Series, together with $3 B' + 4 B''$, compose the second Column: But $3 B' = \frac{3}{4}$, and $4 B'' = 0$, as appears from the Table. Consequently the sum of the Terms of the second Column is $= \frac{3}{4} + \frac{1}{2}x + \frac{1}{2}a$.

By the same Method of proceeding, it will be found, that the sum of the Terms of the third Column is $= \frac{1}{4}x + \frac{1}{2}a$.

From whence it follows that $y = \frac{3}{4} + \frac{1}{2}x + \frac{1}{2}a + \frac{1}{4}x + \frac{1}{2}a$,
or $y = \frac{3}{4} + \frac{3}{4}x + a$.

In the same manner if we write

$$\begin{aligned} 3 C' &= 3 C' \\ 4 C'' &= 4 C'' \\ 5 C''' &= \frac{5}{2} B'' + \frac{5}{4} D' \\ 6 C'''' &= \frac{6}{2} C''' + \frac{6}{4} D'' \\ 7 C^v &= \frac{7}{2} B'''' + \frac{7}{4} D''' \\ 8 C^v &= \frac{8}{2} B^v + \frac{8}{4} D'''' \\ &\&c. \end{aligned}$$

We shall from thence deduce the Equation $z = \frac{1}{2}y + \frac{1}{2}b$
 $+ \frac{1}{4}v + \frac{1}{2}d$.

B b

So

So likewise in the same manner, if we write

$$\begin{aligned} 3 D' &= 3 D' \\ 4 D'' &= 4 D'' \\ 5 D''' &= \frac{5}{2} C'' + \frac{5}{4} B' \\ 6 D'''' &= \frac{6}{2} C''' + \frac{6}{4} B'' \\ 7 D^v &= \frac{7}{2} D'''' + \frac{7}{4} B''' \\ 8 D^v &= \frac{8}{2} D^v + \frac{8}{4} B'''' \\ &\&c. \end{aligned}$$

Lastly, if after the same manner we write

$$\begin{aligned} 3 A' &= 3 A' \\ 4 A'' &= 4 A'' \\ 5 A''' &= \frac{5}{2} D' + \frac{5}{4} C' \\ 6 A'''' &= \frac{6}{2} D'' + \frac{6}{4} C'' \\ 7 A^v &= \frac{7}{2} D''' + \frac{7}{4} C''' \\ 8 A^v &= \frac{8}{2} D^v + \frac{8}{4} C'''' \\ &\&c. \end{aligned}$$

We shall deduce the two following Equations, viz.

$$v = \frac{1}{2} z + \frac{1}{2} c + \frac{1}{4} y + \frac{1}{2} b. \text{ And } x = \frac{1}{2} v + \frac{1}{2} d + \frac{1}{4} z + \frac{1}{2} c.$$

Now the foregoing Equations being Solved, and the values of b, c, d, a restored, it will be found that $y = \frac{45536}{22201}$

$$z = \frac{38724}{22201}, v = \frac{37600}{22201}, x = \frac{33547}{22201}.$$

From which we may conclude, that the several Expectations of B, C, D, A are respectively, First, $4 \times \frac{56}{149} + \frac{45536}{22201} p$.

Secondly, $4 \times \frac{36}{149} + \frac{38724}{22201} p$. Thirdly, $4 \times \frac{32}{149} + \frac{37600}{22201} p$.

Fourthly, $4 \times \frac{25}{149} + \frac{33547}{22201} p$.

The

The Expectations of the Gamesters being thus found, it will be necessary to find the Risks of their being Fined, or otherwise what sum each of them ought justly to give to have their Fines Insured. In order to which, let us form so many Schemes for expressing the Probabilities of the Fines as are sufficient to find the Law of their Process.

And *First*, we may observe, that upon the supposition of *B* beating *A* the first Game, in consequence of which *A* is to be Fined, *B* and *C* have one Chance each for being Fined the second Game, as it thus appears

$$\begin{array}{c|cc} 1 & BA & BA \\ 2 & CB & BC \end{array}$$

Secondly, that *C* has one Chance in four for being Fined the third Game, *B* one Chance likewise, and *D* two; according to the following Scheme,

$$\begin{array}{c|cccc} 1 & BA & BA & BA & BA \\ 2 & CB & CB & BC & BC \\ 3 & DC & CD & DB & BD \end{array}$$

Thirdly, that *D* has two Chances in eight for being Fined the fourth Game, that *A* has three and *C* one; according to the following Scheme,

$$\begin{array}{c|cccccc} 1 & BA & BA & BA & BA & BA & BA \\ 2 & CB & CB & CB & CB & BC & BC \\ 3 & DC & DC & CD & CD & DB & DB \\ 4 & AD & DA & AC & CA & AD & DA \end{array}$$

N. B. The two Combinations *BA*, *BC*, *BD*, *AB*, and *BA*, *BC*, *BD*, *BA* are omitted in this Scheme, as being superfluous; their disposition shewing that the Set must have been ended in three Games, and consequently not affecting the Gamesters as to the Probability of their being Fined the fourth Game; Yet the number of all the Chances must be reckoned as being Eight; since the Probability of any one circumstance is but $\frac{1}{8}$.

These

These Schemes being continued, it will easily be perceived that the circumstances under which the Gamesters find themselves, in respect of their Risks of being Fined, stand related to one another in the same manner as were their Probabilities of Winning; from which consideration a Table of the Risks may easily be composed as follows.

A TABLE of RISKS &c.

		B	C	D	A
I	2	$\frac{1}{2}p$	$\frac{1}{2}p$	-----	-----
II	3	$\frac{1}{4}p$	$\frac{1}{4}p$	$\frac{2}{4}p$	-----
III	4	-----	$\frac{1}{8}p$	$\frac{2}{8}p$	$\frac{3}{8}p$
IIII	5	$\frac{3}{16}p$	$\frac{2}{16}p$	$\frac{2}{16}p$	$\frac{3}{16}p$
V	6	$\frac{6}{32}p$	$\frac{5}{32}p$	$\frac{2}{32}p$	$\frac{3}{32}p$
VI	7	$\frac{6}{64}p$	$\frac{8}{64}p$	$\frac{8}{64}p$	$\frac{4}{32}p$
VII	8	$\frac{7}{128}p$	$\frac{8}{128}p$	$\frac{14}{128}p$	$\frac{13}{128}p$
VIII	9	$\frac{17}{256}p$	$\frac{15}{256}p$	$\frac{14}{256}p$	$\frac{22}{256}p$
&c.	&c.				

Wherefore supposing $B' + B'' + B'''$ &c. $C' + C'' + C'''$ &c. $D' + D'' + D'''$ &c. $A' + A'' + A'''$ &c. to represent the several Probabilities; and supposing that the several sums of these Probabilities are respectively equal to y, x, z, v , we shall have the following Schemes and Equations

$$\begin{aligned}
 B' &= B' \\
 B'' &= B'' \\
 B''' &= \frac{1}{2}A'' + \frac{1}{4}A' \\
 B'''' &= \frac{1}{2}A''' + \frac{1}{4}A'' \\
 B^V &= \frac{1}{2}A'''' + \frac{1}{4}A''' \\
 B^{VI} &= \frac{1}{2}A^{V} + \frac{1}{4}A'''' \\
 &\text{\&c.}
 \end{aligned}$$

$$\text{Hence } y = \frac{3}{4} + \frac{3}{4}x.$$

$$C' =$$

$$\begin{aligned} C' &= C' \\ C'' &= C'' \\ C''' &= \frac{1}{2} B'' + \frac{1}{4} D' \\ C'''' &= \frac{1}{2} B''' + \frac{1}{4} D'' \\ C^v &= \frac{1}{2} B'''' + \frac{1}{4} D''' \\ C^v &= \frac{1}{2} B^v + \frac{1}{4} D'''' \\ &\&c. \end{aligned}$$

$$\text{Hence } z = \frac{1}{2} + \frac{1}{2} y + \frac{1}{4} v.$$

$$\begin{aligned} D' &= D' \\ D'' &= D'' \\ D''' &= \frac{1}{2} C'' + \frac{1}{4} B' \\ D'''' &= \frac{1}{2} C''' + \frac{1}{4} B'' \\ D^v &= \frac{1}{2} C'''' + \frac{1}{4} B''' \\ D^v &= \frac{1}{2} C^v + \frac{1}{4} B'''' \\ &\&c. \end{aligned}$$

$$\text{Hence } v = \frac{1}{4} + \frac{1}{2} z + \frac{1}{4} y.$$

$$\begin{aligned} A' &= A' \\ A'' &= A'' \\ A''' &= \frac{1}{2} D'' + \frac{1}{4} C' \\ A'''' &= \frac{1}{2} D''' + \frac{1}{4} C'' \\ A^v &= \frac{1}{2} D'''' + \frac{1}{4} C''' \\ A^v &= \frac{1}{2} D^v + \frac{1}{4} C'''' \\ &\&c. \end{aligned}$$

$$\text{Hence } x = \frac{1}{2} v + \frac{1}{4} z.$$

The foregoing Equations being resolved, we shall have

$$y = \frac{243}{149}, z = \frac{252}{149}, v = \frac{224}{149}, x = \frac{175}{149}.$$

Let every one of those Fractions be now multiplied by p , and the Products $\frac{243}{149}p$, $\frac{252}{149}p$, $\frac{224}{149}p$, $\frac{175}{149}p$ will express the respective Risks of B, C, D, A , or the sums they might justly give to have their Fines Insured.

But if from the several Expectations of the Gamesters there be subtracted, *First*, the sums by them deposited in the beginning of the Play, and *Secondly*, the Risks of their Fines, there will remain the clear Gain or Loss of each. Wherefore

$$\begin{array}{rcl} \text{From the Expectations of } B & = & 4 \times \frac{56}{149} + \frac{45536}{22201}p, \\ \text{Subtracting his own Stake} & = & \text{I} \\ \text{and also the sum of the Risks} & = & \frac{243}{149}p, \\ \hline \text{There remains his clear Gain} & = & \frac{75}{149} + \frac{9329}{22201}p. \end{array}$$

$$\begin{array}{rcl} \text{From the Expectations of } C & = & 4 \times \frac{36}{149} + \frac{38724}{22201}p, \\ \text{Subtracting his own Stake} & = & \text{I} \\ \text{and also the Sum of his Risks} & = & \frac{252}{149}p, \\ \hline \text{There remains his clear Gain} & = & -\frac{5}{149} + \frac{1176}{22201}p. \end{array}$$

$$\begin{array}{rcl} \text{From the Expectations of } D & = & 4 \times \frac{32}{149} + \frac{37600}{22201}p, \\ \text{Subtracting his own Stake} & = & \text{I} \\ \text{and also the sum of his Risks} & = & + \frac{224}{149}p, \\ \hline \text{There remains his clear Gain} & = & -\frac{21}{149} + \frac{4224}{22201}p. \end{array}$$

$$\begin{array}{rcl} \text{From the Expectations of } A & = & 4 \times \frac{25}{149} + \frac{33547}{22201}p, \\ \text{Subtracting his own Stake} & = & \text{I} \\ \text{and also the Sum of his Risks} & = & \frac{175}{194}p, \\ \text{Lastly, the Fine due to the Stock} & \} & \\ \text{by the loss of the first Game.} & \} & p, \end{array}$$

$$\begin{array}{rcl} \text{There remains his clear Gain} & = & -\frac{49}{149} - \frac{14729}{22201}p. \end{array}$$

The

The foregoing Calculation being made upon the supposition of *B* beating *A* in the beginning of the Play, which supposition neither affects *C* nor *D*, it follows that the sum of the Gains between *B* and *A* ought to be divided equally; and their several Gains will stand as follows,

$$\begin{array}{rcl}
 \text{Gain of } \left\{ \begin{array}{l} A = \frac{13}{149} - \frac{2700}{22201} p, \\ B = \frac{13}{149} - \frac{2700}{22201} p, \\ C = -\frac{1}{149} + \frac{1176}{22201} p, \\ D = -\frac{21}{149} + \frac{4224}{22201} p, \end{array} \right. & & \\
 \text{Sum of the Gains} & = & 0 \quad 0
 \end{array}$$

If $\frac{13}{149} - \frac{2700}{22201} p$, which is the Gain of *A* or *B*, be made = 0; then *p* will be found = $\frac{1937}{2700}$: From which it follows, that if each Man's Stake be to the Fine in the proportion of 2700 to 1937, then *A* and *B* are in this case neither winners nor losers; but *C* wins $\frac{1}{225}$, which *D* loses.

And in the like manner may be found what the proportion between the Stake and the Fine ought to be, to make *C* or *D* play without Advantage or Disadvantage; and also what this proportion ought to be, to make them play with any Advantage or Disadvantage given.

Corollary I. A spectator *R* might at first in consideration of the Sum $4+7p$ paid him in hand, undertake to furnish the four Gamesters with Stakes, and to pay all their Fines.

Corollary II. If the proportion of Skill between the Gamesters be given, then their Gain or Loss may be determined by the methods used in this and the preceding Problem.

Corollary III. If there be never so many Gamesters playing on the conditions of this Problem, and the proportion of Skill between them all be supposed equal, then the Probabilities of winning, or of being Fined, may be determined as follows.

Let

Let $\overline{B'}$, $\overline{C'}$, $\overline{D'}$, $\overline{E'}$, $\overline{F'}$, $\overline{A'}$ denote the Probabilities which B, C, D, E, F, A have of winning the Set, or of being Fined, in any number of Games; and let the Probabilities of winning or being Fined in any number of Games less by one than the preceding, be denoted by $\overline{B''}$, $\overline{C''}$, $\overline{D''}$, $\overline{E''}$, $\overline{F''}$, $\overline{A''}$: And so on.

Then I say that,

$$\begin{aligned}\overline{B'} &= \frac{1}{2} \overline{A''} + \frac{1}{4} \overline{A'''} + \frac{1}{8} \overline{A''''} + \frac{1}{16} \overline{A'''} \\ \overline{C'} &= \frac{1}{2} \overline{B''} + \frac{1}{4} \overline{B'''} + \frac{1}{8} \overline{B''''} + \frac{1}{16} \overline{B'''} \\ \overline{D'} &= \frac{1}{2} \overline{C''} + \frac{1}{4} \overline{C'''} + \frac{1}{8} \overline{C''''} + \frac{1}{16} \overline{C'''} \\ \overline{E'} &= \frac{1}{2} \overline{D''} + \frac{1}{4} \overline{D'''} + \frac{1}{8} \overline{D''''} + \frac{1}{16} \overline{D'''} \\ \overline{F'} &= \frac{1}{2} \overline{E''} + \frac{1}{4} \overline{E'''} + \frac{1}{8} \overline{E''''} + \frac{1}{16} \overline{E'''} \\ \overline{A'} &= \frac{1}{2} \overline{F''} + \frac{1}{4} \overline{F'''} + \frac{1}{8} \overline{F''''} + \frac{1}{16} \overline{F'''}\end{aligned}$$

Corollary IV. If the Terms A, B, C, D, E, F &c. of a Series be continually decreasing, and that the Relation which each Term of the Series has to the same number of preceding ones be constantly express'd by the same number of given Fractions $\frac{1}{p}, \frac{1}{q}, \frac{1}{r}, \frac{1}{s}$ &c. For Example, if E be equal to $\frac{1}{p} D + \frac{1}{q} C + \frac{1}{r} B$, and F be also equal to $\frac{1}{p} E + \frac{1}{q} D + \frac{1}{r} C$, and so on: Then I say that all the Terms *Ad infinitum* of such Series as this, may be easily summed up, by following the steps of the Analysis used in this Problem; of which several Instances will be given in the Problem relating to the duration of Play.

And if the Terms of such Series be multiplyed respectively by any Series of Terms, whose last differences are equal, then the Series resulting from this multiplication is exactly summable.

And if there be two such Series or more, and the Terms of one be respectively multiplyed by the corresponding Terms of the other, then the Series resulting from this multiplication will be exactly summable.

Lastly, If there be several Series so related to one another, that each Term in the one may have to a certain number of terms in the other certain given proportions, and that the order of these proportions be constant and uniform, then will all those Series be exactly summable. The

The foregoing Problem having been formerly Solved by me, and Printed in the *Philosophical Transactions* N° 341. Dr. *Brook Taylor*, that Excellent Mathematician, Secretary to the *Royal Society*, and my Worthy Friend, soon after communicated to me a very Ingenious Method of his, for finding the Relations which the Probabilities of winning bear to one-another, in the case of an equality of Skill between the Gamesters. The Method is as follows.

Let *BACD* represent the four Gamesters; let also the two first Letters represent that *B* beats *A* the first Game, and the other two the order of Play.

This being supposed, the circumstances of the Gamesters will be represented in the next Game by *BCDA* or *CBDA*.

Again, the two preceding Combinations will each of them produce two more Combinations for the Game following, so that the Combinations for that Game will be four in all, viz. *BDAC*, *DBAC*, and *CDAB*, *DCAB*; which may be fitly represented by the following Scheme.



It appears from this Scheme, that if the Combination *CBDA* Happens, which must be in the second Game, then *B* will be in the same Circumstance wherein *A* was the Game before; the conformity of which Circumstances lies in this, that *B* is beat by one who was just come into Play when he engaged him. It appears likewise, that if the Combination *DBAC* Happens, which must be in the third Game, then *B* is again in the same Circumstance wherein *A* was two Games before.

But the Probability of the first Circumstance is $\frac{1}{2}$, and the Probability of the second is $\frac{1}{4}$.

Wherefore the Probability which *B* has of winning the Set, in any number of Games taken from the beginning, is $\frac{1}{2}$ of the Probability which *A* has of winning it in the same number of Games wanting one, taken from the beginning; as
D d also

also $\frac{1}{4}$ of the Probability which he has of winning in the same number of Games wanting two. From which it follows, that if the Probability which *B* has of winning the Set, in Five Games for instance, and the Probabilities which *A* has of winning it in Four and Three, be respectively denoted by B'' , A'''' , A''' , we shall have the Equation, $B'' = \frac{1}{2} A'''' + \frac{1}{4} A'''$, which is conformable to what we had found before. And from the Inspection of the same Scheme may likewise be deduced the Relations of the Probabilities of winning, as they lye between the other Gamesters. And other Schemes of this nature for any number of Gamesters may easily be made in imitation of this, by which the Probabilities of winning or being Fined may be determined by bare Inspection.

P R O B L E M XXXIII.

TWO Gamesters *A* and *B*, whose proportion of Skill is as *a* to *b*, each having a certain number of Pieces, play together on condition that as often as *A* Wins a Game, *B* shall give him one Piece, and that as often as *B* Wins a Game, *A* shall give him one Piece; and that they cease not to Play till such time as either one or the other has got all the Pieces of his adversary. Now let us suppose two Spectators *R* and *S* to lay a Wager about the Ending of the Play, the first of them laying that the Play will be Ended in a certain number of Games which he assigns; the other laying to the contrary. What is the Probability that *S* has of Winning his Wager?

S O L U T I O N.

C A S E I.

LET Two be the number of Pieces which each Gamester has, let also Two be the number of Games about which the Wager is laid: Now because two is the number of Games contended for, let $a + b$ be raised to its Square, viz. $aa + 2ab + bb$; and it is plain that the Term $2ab$ favours *S*, and that the other two are against him, and consequently that the Probability he has of Winning is $\frac{2ab}{a+b}$.

C A S E II.

C A S E II.

LET Two be the number of Pieces of each Gamester, but let Three be the number of Games upon which the Wager is laid: Then $a+b$ being raised to its Cube viz. $a^3 + 3aab + 3abb + b^3$, it is plain that the two Terms a^3 and b^3 are contrary to S , since they denote the number of Chances for winning three times together; 'tis plain also that the other Terms $3aab + 3abb$ are partly for him, partly against him. Let these Terms therefore be divided into their proper parts, viz. $3aab$ into aab , aba , baa , and $3abb$ into abb , bab , baa . Now out of these Six parts there are four which are favourable to S , viz. aba , baa , abb , bab or $2aab + 2abb$; from whence it follows that the Probability which S has of winning his Wager will be $\frac{2aab+2abb}{a+b^3}$: Or dividing both Numerator and Denominator by $a+b$, it will be found to be $\frac{2ab}{a+b}$, which is the same as before.

C A S E III.

LET Two be the number of Pieces of each Gamester, and Four the number of Games upon which the Wager is laid: Let therefore $a+b$ be raised to the fourth Power, which is $a^4 + 4a^3b + 6aabb + 4ab^3 + b^4$. The Terms $a^4 + 4a^3b + 4ab^3 + b^4$ are wholly against S , and the only Term $6aabb$ is partly for him, partly against him: Let this Term therefore be divided into its Parts, viz. $aabb$, $abab$, $abba$, $baab$, $baba$, $baaa$; and Four of these Parts $abab$, $abba$, $baab$, $baba$, or $4aabb$ will be found to favours S ; from which it follows, that his Probability of winning will be $\frac{4aabb}{a+b^4}$.

C A S E IV.

IF Two be the number of Pieces of each Gamester, and Five the number of Games about which the Wager is laid; the Probability which S has of Winning his Wager will be found to be the same as in the preceding Case, viz. $\frac{4aabb}{a+b^4}$.

G E N E.

G E N E R A L L Y.

Let Two be the number of Pieces of each Gamester, and $2 + d$ the number of Games about which R and S contend, and it will be found that the Probability which S has of Winning will be $\frac{2ab^{1+\frac{1}{2}d}}{a+b^{2+d}}$. But if d be an odd Number, substitute $d-1$ in the room of it.

C A S E V.

LET Three be the number of Pieces of each Gamester, and $3 + d$ the number of Games upon which the Wager is laid; and the Probability which S has of Winning will be $\frac{3ab^{1+\frac{1}{2}d}}{a+b^{2+d}}$. But if d be an Odd Number, you are to substitute $d-1$ in the room of it.

C A S E VI.

IF the number of Pieces of each Gamester be more than Three, the Expectation of S , or the Probability there is that the Play will not be Ended in a given number of Games, may be determined in the following manner.

A General R U L E for Determining what Probability there is that the Play will not be Ended in a given number of Games.

LET n be the number of Pieces of each Gamester; let also $n + d$ be the number of Games given. Raise $a+b$ to the Power n , then cut off the two extream Terms, and multiply the remainder by $aa + 2ab + bb$: then cut off again the two Extreams, and multiply again the remainder by $aa + 2ab + bb$, still rejecting the two Extreams, and so on, making as many Multiplications as there are Units in $\frac{1}{2}d$; Let the last Product be the Numerator of a Fraction whose Denominator is $a+b^{n+d}$, and that Fraction will express the Proba-

Probability required, or the Expectation of *S*. Still observing that if *d* be an Odd Number, you write $d-1$ in the room of it.

EXAMPLE I.

LET Four be the number of Pieces of each Gamester, and Ten the number of Games given: In this Case $n = 4$, and $n + d = 10$. Wherefore $d = 6$, and $\frac{1}{2}d = 3$. Let therefore $a+b$ be raised to the Fourth Power, and rejecting continually the Extrems, let three Multiplications be made by $aa + 2ab + bb$. thus,

$$\begin{array}{r}
 a^4 + 4a^3b + 6a^2bb + 4ab^3 + b^4 \\
 aa + 2ab + bb \\
 \hline
 4a^5b + 6a^4bb + 4a^3b^3 \\
 + 8a^4bb + 12a^3b^3 + 8a^2b^4 \\
 + 4a^3b^3 + 6a^2b^4 (+ 4ab^5) \\
 \hline
 14a^4bb + 20a^3b^3 + 14a^2b^4 \\
 aa + 2ab + bb \\
 \hline
 14a^6bb + 20a^5b^3 + 14a^4b^4 \\
 + 28a^5b^3 + 40a^4b^4 + 28a^3b^5 \\
 + 14a^4b^4 + 20a^3b^5 (+ 14a^2b^6) \\
 \hline
 48a^5b^3 + 68a^4b^4 + 48a^3b^5 \\
 aa + 2ab + bb \\
 \hline
 48a^7b^3 + 68a^6b^4 + 48a^5b^5 \\
 + 96a^6b^4 + 136a^5b^5 + 96a^4b^6 \\
 + 48a^5b^5 + 68a^4b^6 (+ 48a^3b^7) \\
 \hline
 164a^6b^4 + 232a^5b^5 + 164a^4b^6.
 \end{array}$$

Wherefore the Probability that the Play will not be ended in Ten Games will be $\frac{164a^6b^4 + 232a^5b^5 + 164a^4b^6}{(a+b)^{10}}$, which expression will be reduced to $\frac{560}{1024}$ or $\frac{35}{64}$, if there be an equality of Skill between the Gamesters. Now this Fraction being subtracted from Unity, the remainder will be $\frac{29}{64}$, which will express the Probability of the Play Ending in

E c Ten

Ten Games: And consequently it is 35 to 29, that two equal Gamesters playing together, there will not be Four Stakes lost on either side in Ten Games.

N. B. The foregoing Operation may be very much contracted by omitting the Letters a and b , and restoring them after the last Multiplication; which may be done in this manner. Make $n + \frac{1}{2}d - 1 = p$, and $\frac{1}{2}d + 1 = q$: Then annex to the respective Terms resulting from the last Multiplication the literal Products $a^p b^q$, $a^{p-1} b^{q+1}$, $a^{p-2} b^{q+2}$ &c. Thus in the foregoing Example, instead of the first Multiplicand $4a^3b + 6aabb + 4ab^3$, we might have taken only $4 + 6 + 4$, and instead of Multiplying Three times by $aa + 2ab + bb$, we might have Multiplied only by $1 + 2 + 1$, which would have made the last Terms to have been $164 + 232 + 164$. Now since that n is $= 4$ and $d = 6$, p will be $= 6$, and $q = 4$; and consequently the literal Products to be annex to the Terms $164 + 232 + 164$ will be respectively $a^6 b^4$, $a^5 b^5$, $a^4 b^6$, which will make the Terms resulting from the last Multiplication to be $164a^6 b^4 + 232a^5 b^5 + 164a^4 b^6$, as they had been found before.

EXAMPLE II.

LET Five be the number of Pieces of each Gamester, and Ten the number of Games given. Let also the proportion of Skill between A and B be as Two to One.

Since n is $= 5$, and $n + d = 10$, it follows that d is $= 5$: Now d being an odd number must be lessened by Unity, and supposed $= 4$, so that $\frac{1}{2}d = 2$. Let therefore $a + b$ be raised to the fifth Power; and always rejecting the extreams, Multiply twice by $aa + 2ab + bb$, or rather by $1 + 2 + 1$; thus,

$\begin{array}{r} 1) + 5 + 10 + 10 + 5 (+1 \\ \hline 1 + 2 + 1 \end{array}$	$\begin{array}{r} 20 + 35 + 35 + 20 \\ \hline 1 + 2 + 1 \end{array}$
$\begin{array}{r} 5) + 10 + 10 + 5 \\ + 10 + 20 + 20 + 10 \\ + 5 + 10 + 10 (+5 \\ \hline 20 + 35 + 35 + 20 \end{array}$	$\begin{array}{r} 20) + 35 + 35 + 20 \\ 40 + 70 + 70 + 40 \\ 20 + 35 + 35 (+20 \\ \hline 75 + 125 + 125 + 75 \end{array}$
$\hline 20 + 35 + 35 + 20$	$\hline 75 + 125 + 125 + 75$
	Now

Now to supply the literal Products that are wanting, let $n + \frac{1}{2}d - 1$ be made $= p$, and $\frac{1}{2}d + 1 = q$, then p will be $= 6$ and $q = 3$. Wherefore the Products to be annexed, viz. $a^p b^q$, $a^{p-1} b^{q+1}$ &c. will become $a^6 b^3$, $a^5 b^4$, $a^4 b^5$, $a^3 b^6$; and consequently the Expectation of S will be found to be

$$\frac{75 a^6 b^3 + 125 a^5 b^4 + 125 a^4 b^5 + 75 a^3 b^6}{a+b}.$$

N. B. When n is an odd number, as it is in this Case, the Expectation of S will always be divisible by $a+b$. Wherefore dividing both Numerator and Denominator by $a+b$, the foregoing Expression will be reduced to

$$\frac{75 a^5 b^3 + 50 a^4 b^4 + 75 a^3 b^5}{a+b}, \text{ or } 25 a^3 b^3 \times \frac{3aa + 2ab + 3bb}{a+b}.$$

Let now a be interpreted by 2 and b by 1, and the Expectation of S will become $\frac{3800}{6561}$.

PROBLEM XXXIV.

THE same Things being given as in the preceding Problem; to find the Expectation of R , or otherwise what the Probability is that the Play will be Ended in a given number of Games.

SOLUTION.

First, It is plain that if the Expectation of S , obtained by the preceding Problem, be subtracted from Unity, there will remain the Expectation of R .

Secondly, Since the Expectation of S decreases continually as the number of Games increases, and that the Terms we rejected in the former Problem being divided by $aa + 2ab + bb$ are the Decrement of his Expectation; it follows, that if those rejected Terms be divided continually by $a+b$ they will be the Increment of the Expectation of R . Wherefore the Expectation of R may be expressed by means of those rejected Terms. Thus, in the second Example of the preceding Problem, the Expectation of R expressed by means of the rejected Terms will be found to be

$$a^5 + b^5$$

$$\frac{a^5 + b^5}{a+b} + \frac{5a^4b + 5ab^4}{a+b} + \frac{20a^3b^2 + 20a^2b^3}{a+b} \quad \text{or}$$

$$\frac{a^5 + b^5}{a+b} \times 1 + \frac{5ab}{a+b} + \frac{20aabb}{a+b}.$$

In the like manner, if Six were the number of the Pieces of each Gamester, and the number of Games were Fourteen; it would be found that the Expectation of R would be

$$\frac{a^6 + b^6}{a+b} \times 1 + \frac{6ab}{a+b} + \frac{27aabb}{a+b} + \frac{110a^3b^3}{a+b} + \frac{429a^4b^4}{a+b} :$$

And if Seven were the number of the Pieces of each Gamester, and the number of Games given were Fifteen; then the Expectation of R would be found to be

$$\frac{a^7 + b^7}{a+b} \times 1 + \frac{7ab}{a+b} + \frac{35aabb}{a+b} + \frac{154a^3b^3}{a+b} + \frac{637a^4b^4}{a+b}.$$

N. B. The number of Terms of these Series will always be equal to $\frac{1}{2}d + 1$, if d be an even number, or to $\frac{d+1}{2}$ if it be odd.

Thirdly, All the Terms of these Series have to one another certain Relations; which being once discovered, each Term of any Series resulting from any Case of this Problem, may be easily generated from the preceding ones.

Thus in the first of the two last foregoing Series, the Numerical Coefficient belonging to the Numerator of each Term, may be derived from the preceding ones, in the following manner. Let K , L , M be the Three last Coefficients, and let N be the Coefficient of the next Term wanted; then it will be found that N in that Series will constantly be equal to $6M - 9L + 2K$. Wherefore if the Term which would follow $\frac{429a^4b^4}{a+b}$, in the Case of Sixteen Games given, were desired; then make $M = 429$, $L = 110$, $K = 27$, and the following Coefficient will be found 1638. From whence it appears that the Term it self would be $\frac{1638a^5b^5}{a+b}$.

Likewise, in the second of the two foregoing Series, if the Law by which each Term is related to the preceding ones were

were demanded, it might be thus found. Let K, L, M , be the Coefficients of the three last Terms, and N the Coefficient of the Term desired; then N will in that Series, constantly be equal to $7M - 14L + 7K$, or to $\overline{M - 2L + K} \times 7$. Now this Coefficient being obtained, the Term to which it belongs is formed immediately.

But if the general Law, by which each Coefficient is generated from the preceding ones, be demanded, it will be exprest as follows. Let n be the number of Pieces of each Gamester: Then each Coefficient contains

$$\begin{aligned} & n \text{ times the last,} \\ - & n \times \frac{n-3}{2} \text{ times the last but one,} \\ + & n \times \frac{n-4}{2} \times \frac{n-5}{3} \text{ times the last but two,} \\ - & n \times \frac{n-5}{2} \times \frac{n-6}{3} \times \frac{n-7}{4} \text{ times the last but three,} \\ + & n \times \frac{n-6}{2} \times \frac{n-7}{3} \times \frac{n-8}{4} \times \frac{n-9}{5} \text{ times the last but four,} \\ & \&c. \end{aligned}$$

Thus the number of Pieces of each Gamester being Six, the first Term n would be = 6, the second Term $n \times \frac{n-3}{2}$ would be = 9, the third Term $n \times \frac{n-4}{2} \times \frac{n-5}{3}$ would be = 2; the rest of the Terms vanishing in this Case. Wherefore if K, L, M are the three last Coefficients, the Coefficient of the following Term will be $6M - 9L + 2K$.

Fourthly, The Coefficient of any Term of these Series may be found, independently from any relation they may have to the preceding ones: In order to which it is to be observed that each Term of these Series is proportional to the Probability of the Plays Ending in a certain number of Games precisely: Thus in the Series which expresses the Expectation of R , when each Gamester is supposed to have Six Pieces, *viz.*

$$\frac{a^6+b^6}{a+b} \times 1 + \frac{6ab}{a+b^2} + \frac{27aabb}{a+b^4} + \frac{110a^3b^3}{a+b^6} + \frac{429a^4b^4}{a+b^8},$$

the last Term, being multiplied by the common Multiplier $\frac{a^6+b^6}{a+b}$ set down before the Series, that is the Pro-

duct $\frac{429a^4b^4 \times a^6+b^6}{a+b}$, denotes the Probability of the Plays

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Ending in Fourteen Games precisely. Wherefore if that Term were desired which expresses the Probability of the Plays Ending in Twenty Games precisely, or in any number of Games denoted by $n+d$, I say that the Coefficient of that Term will be,

$\frac{1}{1} \times \frac{n}{1} \times \frac{n+d-1}{2} \times \frac{n+d-2}{3} \times \frac{n+d-3}{4}$ &c. continued to so many Terms as there are Units in $\frac{1}{2}d + 1$.

— $\frac{1}{1} \times \frac{3n}{1} \times \frac{n+d-1}{2} \times \frac{n+d-2}{3} \times \frac{n+d-3}{4}$ &c. continued to so many Terms as there are Units in $\frac{1}{2}d + 1 - n$.

+ $\frac{1}{1} \times \frac{5n}{1} \times \frac{n+d-1}{2} \times \frac{n+d-2}{3} \times \frac{n+d-3}{4}$ &c. continued to so many Terms as there are Units in $\frac{1}{2}d + 1 - 2n$.

— $\frac{1}{1} \times \frac{7n}{1} \times \frac{n+d-1}{2} \times \frac{n+d-2}{3} \times \frac{n+d-3}{4}$ &c. continued to so many Terms as there are Units in $\frac{1}{2}d + 1 - 3n$. &c.

Let now $n+d$ be supposed = 20, n being already supposed = 6, then the Coefficient demanded will be found from the general Rule to be,

$$\begin{aligned} & \frac{1}{1} \times \frac{6}{1} \times \frac{19}{2} \times \frac{18}{3} \times \frac{17}{4} \times \frac{16}{5} \times \frac{15}{6} \times \frac{14}{7} = 23256 \\ & - \frac{1}{1} \times \frac{18}{1} = 18 \end{aligned}$$

Wherefore the Coefficient demanded will be $23256 - 18 = 23238$: And then the Term it self to which this Coefficient does belong, will be $\frac{23238 a^7 b^7}{a+b^{14}}$. Consequently the Probability of the Plays Ending in Twenty Games precisely will be $\frac{a^6 + b^6}{a+b^6} \times \frac{23238 a^7 b^7}{a+b^{14}}$.

Fifthly, By the help of the two Methods explained in this Problem (whereof the first is for finding the relation which any Term of the Series resulting from the Problem, has to a certain number of preceding ones; and the second for finding any Term of the Series independently from any other Term) together with the Method of summing up any given num-

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number of Terms of these Series, (which shall be explained in its place); the Probability of the Plays Ending in any given number of Games, will be found much more readily than can be done by either of the two first Methods taken singly.

PROBLEM XXXV.

Supposing A and B to play together, till such time as Four Stakes are Won or Lost on either side: What must be their proportion of Skill, to make it as Probable that the Play will be Ended in Four Games, as not?

SOLUTION.

THE Probability of the Play Ending in Four Games, is by the preceding Problem $\frac{a^4 + b^4}{a^4 + b^4}$; Now because, by *Hypothesis*, it is to be an equal Chance whether the Play Ends or Ends not in Four Games; let this expression of the Probability be made equal to $\frac{1}{2}$; And we shall have this Equation $\frac{a^4 + b^4}{a^4 + b^4} = \frac{1}{2}$, which, making $b, a :: 1, z$, is reduced to $\frac{z^4 + 1}{z^4 + 1} = \frac{1}{2}$, or $z^4 - 4z^3 - 6zz - 4z + 1 = 0$. Let $12zz$ be added on both sides the Equation, then will $z^4 - 4z^3 + 6zz - 4z + 1 = 12zz$; and extracting the square Root on both sides, it will be reduced to this *Quadratick* Equation $zz - 2z + 1 = z\sqrt{12}$, whose double Root is $z = 5,274$ and $\frac{1}{5,274}$. Wherefore whether the Skill of *A* be to that of *B* as 5.274 to 1 , or as 1 to 5.274 , there will be an equality of Chance for the Play to be Ended or not Ended in Four Games.

PROBLEM XXXVI.

Supposing that A and B Play till such time as Four Stakes are Won or Lost: What must be their proportion of Skill, to make it a Wager of Three to One, that the Play will be Ended in Four Games?

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SOLUTION.

THE Probability of the Plays Ending in Four Games, arising from the number of Games Four, from the number of Stakes Four, and from the proportion of Skill is $\frac{a^4+b^4}{a+b^4}$. The same Probability arising from the odds of Three to One, is $\frac{3}{4}$: Wherefore $\frac{a^4+b^4}{a+b^4} = \frac{3}{4}$, and supposing $b, a :: 1, z$, the foregoing Equation will be changed into $\frac{z^4+1}{z+1^4} = \frac{3}{4}$, or $z^4 - 12z^3 - 18zz - 12z + 1 = 0$. Let $56zz$ be added on both sides the Equation, then we shall have $z^4 - 12z^3 + 38zz - 12z + 1 = 56zz$. And Extracting the square Root on both sides, we shall have $zz - 6z + 1 = z\sqrt{56}$, the Roots of which Equation will be found 13.407 and $\frac{1}{13.407}$. Wherefore, whether the Skill of A be to that of B as 13.407 to 1 , or as 1 to 13.407 , 'tis a Wager of Three to One, that the Play will be ended in Four Games.

PROBLEM XXXVII.

Supposing that A and B Play till such time as Four Stakes are Won or Lost; What must be their proportion of Skill, to make it an equal Wager that the Play will be Ended in Six Games?

SOLUTION.

THE Probability of the Plays Ending in Six Games, arising from the given number Six, from the number of Stakes Four, and from the proportion of Skill, is $\frac{a^4+b^4}{a+b^4} \times 1 + \frac{4ab}{a+b^2}$. The same Probability arising from an equality of Chance for Ending or not Ending in Six Games, is equal to $\frac{1}{2}$, from whence results the Equation $\frac{a^4+b^4}{a+b^4} \times 1 + \frac{4ab}{a+b^2} = \frac{1}{2}$ which by making $b, a :: 1, z$ may be changed into the following, viz. $z^6 + 6z^5 - 13z^4 - 20z^3 - 13zz + 6z + 1 = 0$. In

In this Equation, the Coefficients of the Terms equally distant from the Extrems being the same, let it be supposed that the Equation is generated from the Multiplication of two other Equations of the same nature, viz. $zz - yz + 1 = 0$, and $z^4 + pz^3 + qzz + pz + 1 = 0$. Now the Equation resulting from the Multiplication of these two will be

$$\begin{array}{ccccccc} & -y & + 1 & & 2p & + 1 & + p \\ z^6 & & & z^5 - pyz^4 & + & z^3 - pyzz & z + 1 = 0, \\ & + p & + q & - qy & + q & - y & \end{array}$$

which being compared with the first Equation, we shall have $p - y = 6$, $1 - py + q = -13$, $2p - qy = -20$.

From hence will be deduced a new Equation, viz. $y^3 + 6yy - 16y - 32 = 0$, one of whose Roots will be 2.9644; which being substituted in the Equation $zz - yz + 1 = 0$, we shall at last come to the Equation $zz - 2.9644z + 1 = 0$, of which the two Roots will be 2.576 and $\frac{1}{2.576}$. It follows therefore, that if the Skill of either Gamester be to that of the other as 2.576 to 1; there will be an equal Chance for Four Stakes to be Lost, or not to be Lost, in Six Games.

Corollary. If the Coefficients of the Extream Terms of an Equation, and likewise the Coefficients of the other Terms equally distant from the Extrems, be the same, that Equation will be reducible to another, in which the Dimensions of the highest Term will not exceed half the Dimensions of the highest Term in the former.

PROBLEM XXXVIII.

Supposing A and B, whose proportion of Skill is as a to b, to Play together till such time as A either Wins a certain number q of Stakes, or B some other number p of them: What is the Probability that the Play will not be Ended in a given number of Games?

SOLUTION.

TAKE the Binomial $a+b$, and rejecting continually those Terms in which the Dimensions of the quantity a exceed the Dimensions of the quantity b by q , rejecting also those

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those Terms in which the Dimensions of the quantity b exceed the Dimensions of the quantity a by p ; multiply constantly the remainder by $a+b$, and make as many Multiplications, as there are Units in the given number of Games wanting one. Then shall the last Product be the Numerator of a Fraction expressing the Probability required; the Denominator of which Fraction always being the Binomial $a+b$ raised to that Power which is denoted by the given number of Games.

EXAMPLE.

LET p be $= 3$, $q = 2$, and let the given number of Games be $= 7$. Let the following Operation be made according to the foregoing directions.

$$\begin{array}{r}
 a + b \\
 a + b \\
 \hline
 aa) + 2ab + bb \\
 a + b \\
 \hline
 2aab + 3abb + (+b^3) \\
 a + b \\
 \hline
 2a^3b) + 5aabb + 3ab^3 \\
 a + b \\
 \hline
 5a^3bb + 8aab^3 + (3ab^4) \\
 a + b \\
 \hline
 5a^4bb) + 13a^3b^3 + 8aab^4 \\
 a + b \\
 \hline
 13a^4b^3 + 21a^3b^4 (+ 8aab^5.
 \end{array}$$

From this Operation we may conclude, that the Probability that the Play will not be Ended in Seven Games is equal to $\frac{13a^4b^3 + 21a^3b^4}{a+b^7}$. Now if an Equality of Skill be supposed between A and B , the Expression of this Probability

bility will be reduced to $\frac{13+21}{128}$ or $\frac{17}{64}$: Wherefore the Probability that the Play will End in Seven Games will be $\frac{47}{64}$; from which it follows that 'tis 47 to 17 that in Seven Games, either *A* wins two Stakes or *B* wins three.

PROBLEM XXXIX.

THE same Things being suppos'd as in the preceding Problem, to find the Probability of the Plays being Ended in a given number of Games.

SOLUTION.

First, If the Probability of the Plays not being Ended in the given number of Games be subtracted from Unity, there will remain the Probability of its being Ended in the same number of Games.

Secondly, This Probability may be expressed by means of the Terms rejected in the Operation belonging to the preceding Problem; Thus, if the number of Stakes be Three and Two, the Probability of the Plays being Ended in Seven Games may be expressed as follows.

$$\begin{aligned} & \frac{aa}{(a+b)^2} \times 1 + \frac{2ab}{(a+b)^2} + \frac{5aabb}{(a+b)^4} \\ & + \frac{b^3}{(a+b)^3} \times 1 + \frac{3ab}{(a+b)^2} + \frac{8aabb}{(a+b)^4} \end{aligned}$$

Supposing *a* and *b* both equal to Unity, the sum of the first Series will be $= \frac{29}{64}$, and the sum of the second will be $= \frac{18}{64}$; which two sums being added together, the aggregate $\frac{47}{64}$ expresses the Probability that in Seven Games either *A* shall win Two Stakes or *B* Three.

Thirdly, The Probability of the Plays being Ended in a certain number of Games, or sooner, is always composed of a double Series, when the Stakes are unequal; which double Series is reduced to a single one, in the Case of equality of Stakes.

The first Series always expresses the Probability there is that *A*, in a given number of Games, or sooner, may win of *B* the num-

number q of Stakes, excluding the Probability there is, that B , before that time, may be in a circumstance of winning the number p of Stakes. Both which Probabilities are not inconsistent together; for A in Fifteen Games, for Instance, or sooner, may win Two Stakes of B , though B before that time may have been in a circumstance of winning Three Stakes of A .

The second Series always expresses the Probability there is that B , in that given number of Games or sooner, may win of A a certain number p of Stakes, excluding the Probability there is that A , before that time, may win of B the number q of Stakes.

The first Terms of each Series may be represented respectively by the following Terms.

$$\frac{a^q}{a+b^q} \times 1 + \frac{qab}{a+b^2} + \frac{q \times q + 3 \times aabb}{1 \times 2 \times a + b^4} + \frac{q \times q + 4 \times q + 5 \times a^3b^3}{1 \times 2 \times 3 \times a + b^6} \&c. \text{ and}$$

$$\frac{b^p}{a+b^p} \times 1 + \frac{pab}{a+b^2} + \frac{p \times p + 3 \times aabb}{1 \times 2 \times a + b^4} + \frac{p \times p + 4 \times p + 5 \times a^3b^3}{1 \times 2 \times 3 \times a + b^6} \&c.$$

Each of these Series continuing in that regularity, till such time as there be a number p of Terms taken in the first, and a number q of Terms taken in the second; after which the Law of the continuation breaks off.

Now in order to find any of the Terms following in either of these Series, proceed thus; let $p + q - 2$ be called l ; let the Coefficient of the Term desired be T ; let also the Coefficients of the preceding Terms taken in an inverted order be S, R, Q, P &c. Then will T be equal to $lS - \frac{l-1}{1} \times \frac{l-2}{2} R + \frac{l-2}{1} \times \frac{l-3}{2} \times \frac{l-4}{3} Q - \frac{l-3}{1} \times \frac{l-4}{2} \times \frac{l-5}{3} \times \frac{l-6}{4} P$ &c. Thus if p be $= 3$, and $q = 2$, then l will be $3 + 2 - 2 = 3$. Wherefore $lS - \frac{l-1}{1} \times \frac{l-2}{2} \times R$ would in this Case be equal to $3S - R$; which shews that the Coefficient of any Term desired would be constantly three times the last, *minus* once the last but one.

To apply this, let it be required to find what Probability there is, that in Fifteen Games or sooner, either A shall win two Stakes of B , or B three Stakes of A ; or which is all one, to find what Probability there is, that the Play shall end in Fifteen Games or sooner, A and B resolving to Play, till such time as A either wins three Stakes, or B two.

Let

Let Two and Three, in the two foregoing Series, be substituted respectively in the room of q and p ; then the three first Terms of the first Series will be, setting aside the common Multiplier, $1 + \frac{2ab}{a+b} + \frac{5aabb}{a+b}$: Likewise the two first Terms of the second will be $1 + \frac{3ab}{a+b}$. Now because the Coefficient of any Term desired in each Series, is respectively three times the last, *minus* once the last but one, it follows, that the next Coefficient in the first Series will be 13, and by the same rule the next to it 34, and so on. In the same manner the next Coefficient in the second Series will be found to be 8, and the next to it 21, and so on. Wherefore, restoring the common Multipliers, the two Series will be

$$\begin{aligned} \frac{a^2}{a+b} \times & 1 + \frac{2ab}{a+b} + \frac{5aabb}{a+b} + \frac{13a^3b^3}{a+b} + \frac{34a^4b^4}{a+b} + \frac{89a^5b^5}{a+b} \\ & + \frac{233a^6b^6}{a+b} \\ \frac{b^3}{a+b} \times & 1 + \frac{3ab}{a+b} + \frac{8aabb}{a+b} + \frac{21a^3b^3}{a+b} + \frac{55a^4b^4}{a+b} + \frac{144a^5b^5}{a+b} \\ & + \frac{377a^6b^6}{a+b} \end{aligned}$$

If we suppose an equality of Skill between A and B , the sum of the first Series will be $\frac{18778}{32768}$, the sum of the second will be $\frac{12393}{32768}$, and the aggregate of these two sums will be $\frac{31171}{32768}$, which will express the Probability of the Plays Ending in Fifteen Games or sooner. This last Fraction being subtracted from Unity, there will remain $\frac{1597}{32768}$, which expresses the Probability of the Plays continuing for Fifteen Games at least: Wherefore 'tis 31171 to 1597, or 39 to 2 nearly, that one of the two equal Gamesters, that shall be pitcht upon, shall in Fifteen Games, or sooner, either win Two Stakes of his adversary, or lose Three to him.

N. B. The Index of the Denominator in the last Term of each Series, and the Index of the common Multiplier pre-

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fixt to it, being added together, must either equal the number of Games given, or be less than it by Unity. Thus, in the first Series, the Index 12 of the Denominator of the last Term, and the Index 2 of the common Multiplier being added together, the sum is 14, which is less by Unity than the number of Games given. So likewise in the second Series, the Index 12 of the Denominator of the last Term, and the Index 3 of the common Multiplier being added together, the sum is 15, which precisely equals the number of Games given.

It is carefully to be observed, that these two Series taken together, express the Expectation of one and the same Person, and not of two different Persons; that is properly the Expectation of a spectator who lays a Wager that the Play will be Ended in a given number of Games. Yet in one Case they may express the Expectations of two different Persons: For Instance, of the Gamesters themselves, provided that both Series be continued infinitely; for in that Case, the first Series infinitely continued will express the Probability that the Gamester *A* may sooner win two Stakes of *B*, than that he may lose three to him: Likewise the second Series infinitely continued will express the Probability that the Gamester *B* may sooner win three Stakes of *A*, than that he may lose two to him. And it will be found, when we come to treat of the method of summing up these Series, that the first Series infinitely continued will be to the second infinitely continued, in the proportion of $aa \times aa + ab + bb$ to $b^3 \times a + b$; that is, in the Case of an equality of Skill, as three to two; which is conformable to what we have said in our IXth. Problem.

Fourthly, Any Term of these Series may be found independently from any of the preceding ones: For if a Wager be laid that *A* shall either win a certain number of Stakes denominated by *q*, or that *B* shall win a certain number of them denominated by *p*, and that the number of Games given be expressed by $q + d$; then I say that the Coefficient of any Term in the first Series, answering to that number of Games, will be

$$+ \frac{1}{1} \times \frac{q}{1} \times \frac{q+d-1}{2} \times \frac{q+d-2}{3} \times \frac{q+d-3}{4} \text{ \&c. continued to so many Terms as there are Units in } \frac{1}{2}d + 1.$$

$$= \frac{1}{1}$$

$$- \frac{1}{1} \times \frac{q+2p}{1} \times \frac{q+d-1}{2} \times \frac{q+d-2}{3} \times \frac{q+d-3}{4} \&c.$$
 continued to so many Terms as there are Units in $\frac{1}{2}d + 1 - p$.

$$+ \frac{1}{1} \times \frac{3q+2p}{1} \times \frac{q+d-1}{2} \times \frac{q+d-2}{3} \times \frac{q+d-3}{4} \&c.$$
 continued to so many Terms as there are Units in $\frac{1}{2}d + 1 - p - q$.

$$- \frac{1}{1} \times \frac{3q+4p}{1} \times \frac{q+d-1}{2} \times \frac{q+d-2}{3} \times \frac{q+d-3}{4} \&c.$$
 continued to so many Terms as there are Units in $\frac{1}{2}d + 1 - 2p - q$.

$$+ \frac{1}{1} \times \frac{5q+4p}{1} \times \frac{q+d-1}{2} \times \frac{q+d-2}{3} \times \frac{q+d-3}{4} \&c.$$
 continued to so many Terms as there are Units in $\frac{1}{2}d + 1 - 2p - 2q$.

$$- \frac{1}{1} \times \frac{5q+6p}{1} \times \frac{q+d-1}{2} \times \frac{q+d-2}{3} \times \frac{q+d-3}{4} \&c.$$
 continued to so many Terms as there are Units in $\frac{1}{2}d + 1 - 3p - 2q$.

$$+ \frac{1}{1} \times \frac{7q+6p}{1} \times \frac{q+d-1}{2} \times \frac{q+d-2}{3} \times \frac{q+d-3}{4} \&c.$$
 continued to so many Terms as there are Units in $\frac{1}{2}d + 1 - 3p - 3q$.
 &c.

And the same Law will hold for the other Series, calling $p+d$ the number of Games given, and changing q into p , and p into q , as also d into d .

But when it Happens that d is an odd number, substitute $d-1$ in the room of it, and the like for d .

PROBLEM XL.

IF A and B, whose proportion of Skill is supposed as a to b , play together: What is the Probability that one of them, suppose A, may in a number of Games not exceeding a number given, win of B a certain number of Stakes? Leaving it wholly indifferent whether B before the expiration of those Games may or may not have been in a circumstance of winning the same, or any other number of Stakes of A.

SOLUTION.

SUPposing n to be the number of Stakes which A is to win of B, and $n+d$ the given number of Games; let $a+b$ be raised to the Power whose Index is $n+d$: Then if d be an

an odd Number, take so many Terms of that Power as there are Units in $\frac{d+1}{2}$; take also as many of the Terms next following as have been taken already, but prefix to them, in an inverted order, the Coefficients of the preceding Terms. But if d be an even number, take so many Terms of the said Power as there are Units in $\frac{1}{2}d + 1$; then take as many of the Terms next following as there are Units in $\frac{1}{2}d$, and prefix to them, in an inverted order, the Coefficients of the preceding Terms, omitting the last of them; and those Terms taken all together will compose the Numerator of a Fraction expressing the Probability required, its Denominator being $(a+b)^{n+d}$.

EXAMPLE I.

Supposing the number of Stakes which A is to win to be Three, and the given number of Games to be Ten; let $a+b$ be raised to the tenth Power, viz. $a^{10} + 10a^9b + 45a^8bb + 120a^7b^3 + 210a^6b^4 + 252a^5b^5 + 210a^4b^6 + 120a^3b^7 + 45aab^8 + 10ab^9 + b^{10}$. Then by reason that n is $= 3$ and $n+d = 10$, it follows that d is $= 7$, and $\frac{d+1}{2} = 4$. Wherefore let the Four first Terms of the said Power be taken, viz. $a^{10} + 10a^9b + 45a^8bb + 120a^7b^3$, and let the Four Terms next following be taken likewise, without regard to their Coefficients; then prefix to them, in an Inverted order, the Coefficients of the preceding Terms: Thus the Four Terms following with their new Coefficients, will be $120a^6b^4 + 45a^5b^5 + 10a^4b^6 + 1a^3b^7$. And the Probability which A has of winning Three Stakes of B in Ten Games, or sooner, will be expressed by the following Fraction,

$$\frac{a^{10} + 10a^9b + 45a^8bb + 120a^7b^3 + 120a^6b^4 + 45a^5b^5 + 10a^4b^6 + a^3b^7}{(a+b)^{10}}$$

which, in the Case of an equality of Skill between A and B , will be reduced to $\frac{352}{1024}$ or $\frac{11}{32}$.

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EXAMPLE II.

Supposing the number of Stakes which A is to win to be Four, and the given number of Games to be Ten; let $a+b$ be raised to the tenth Power, and by reason that n is $= 4$, and $n+d = 10$, it follows, that $d = 6$ and $\frac{1}{2}d + 1 = 4$; wherefore let the Four first Terms of the said Power be taken, viz. $a^{10} + 10a^9b + 45a^8bb + 120a^7b^3$; take also Three of the Terms following, but prefix to them, in an inverted order, the Coefficients of the Terms already taken, omitting the last of them. Hence the Three Terms following with their new Coefficients will be $45a^6b^4 + 10a^5b^5 + 1a^4b^6$. And the Probability which A has of winning Four Stakes of B , in Ten Games or sooner, will be expressed by the following Fraction

$$\frac{a^{10} + 10a^9b + 45a^8bb + 120a^7b^3 + 45a^6b^4 + 10a^5b^5 + 1a^4b^6}{a+b^{10}}$$

which, in the Case of an equality of Skill between A and B , will be reduced to $\frac{232}{1024}$ or $\frac{29}{128}$.

Another SOLUTION.

Supposing, as before, that n be the number of Stakes which A is to win, and that the given number of Games be $n+d$ the Probability which A has of winning will be expressed by the following Series, viz.

$$\frac{a^n}{a+b^n} \times 1 + \frac{nab}{a+b^2} + \frac{n \times n + 3 \times aabb}{1 \times 2 \times a+b^4} + \frac{n \times n + 4 \times n + 5 \times a^3b^3}{1 \times 2 \times 3 \times a+b^6} + \frac{n \times n + 5 \times n + 6 \times n + 7 \times a^4b^4}{1 \times 2 \times 3 \times 4 \times a+b^8} \&c.$$

which Series ought to be continued to so many Terms as there are Units in $\frac{1}{2}d + 1$; always observing to substitute $d-1$ in the room of d , in case d be an odd number, or which is the same thing, taking so many Terms as there are Units in $\frac{d+1}{2}$.

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Now

Now supposing, as in the first Example of the preceding Solution, that Three is the number of Stakes, and Ten the given number of Games, as also that there is an equality of Skill between *A* and *B*, the foregoing Series will become

$$\frac{1}{8} \times 1 + \frac{3}{4} + \frac{9}{16} + \frac{28}{64} = \frac{11}{32}, \text{ as before.}$$

R E M A R K.

Monsieur de *Monmort*, in the Second Edition of his Book of Chances, having given a very handsom Solution of the Problem relating to the duration of Play, (which Solution is coincident with that of Monsieur *Nicolas Bernoulli*, to be seen in that Book) and the Demonstration of it being very naturally deduced from our first Solution of the foregoing Problem, I thought the Reader would be well pleased to see it transferred to this place.

Let it therefore be proposed to find the number of Chances there are, for *A* either to win Two Stakes of *B*, or for *B* to win Three of *A* in Fifteen Games.

The number of Chances required is expressed by two Branches of Series; all the Series of the first Branch taken together express the number of Chances there are for *A* to win Two Stakes of *B*, exclusive of the number of Chances there are for *B*, before that time, to win Three Stakes of *A*. All the Series of the second Branch taken together express the number of Chances there are for *B* to win Three Stakes of *A*, exclusive of the number of Chances there are for *A*, before that time, to win Two Stakes of *B*.

First Branch of SERIES.

$$\begin{aligned} & a^{15} a^{14} b \ a^{13} b^2 \ a^{12} b^3 \ a^{11} b^4 \ a^{10} b^5 \ a^9 b^6 \ a^8 b^7 \ a^7 b^8 \ a^6 b^9 \ a^5 b^{10} \ a^4 b^{11} \ a^3 b^{12} \ a^2 b^{13} \\ & 1 + 15 + 105 + 455 + 1365 + 3003 + 5005 + 5005 + 3003 + 1365 + 455 + 105 + 15 + 1 \\ & - 1 - 15 - 105 - 455 - 455 - 105 - 15 - 1 \\ & + 1 + 15 + 15 + 1 \end{aligned}$$

Second Branch of SERIES.

$$\begin{aligned} & b^{15} b^{14} a \ b^{13} a^2 \ b^{12} a^3 \ b^{11} a^4 \ b^{10} a^5 \ b^9 a^6 \ b^8 a^7 \ b^7 a^8 \ b^6 a^9 \ b^5 a^{10} \ b^4 a^{11} \ b^3 a^{12} \\ & 1 + 15 + 105 + 455 + 1365 + 3003 + 5005 + 3003 + 1365 + 455 + 105 + 15 + 1 \\ & - 1 - 15 - 105 - 455 - 1365 - 455 - 105 - 15 - 1 \\ & + 1 + 15 + 1 \end{aligned}$$

The

The literal Quantities, which are commonly annex to the numerical ones, are here written on the top of them; which is done, to the end that each Series being contained in one line, the dependency they have upon one another, may thereby be made the more conspicuous.

The first Series of the first Branch expresses the number of Chances there are for *A* to win Two Stakes of *B*, including the number of Chances there are for *B*, before the expiration of the Fifteen Games, to be in a circumstance of winning Three Stakes of *A*; which number of Chances may be deduced from our foregoing Problem.

The second Series of the first Branch is a part of the first, and expresses the number of Chances there are, for *B* to win Three Stakes of *A*, out of the number of Chances there are for *A* in the first Series, to win Two Stakes of *B*. It is to be observed about this Series, *First*, that the Chances of *B* expressed by it are not restrained to Happen in any Order, that is, either before or after *A* has won Two Stakes of *B*. *Secondly*, that the literal Products belonging to it are the same with those of the corresponding Terms of the first Series. *Thirdly*, that it begins and ends at an interval from the first and last Terms of the first Series equal to the number of Stakes which *B* is to win. *Fourthly*, that the numbers belonging to it are the numbers of the first Series repeated in order, and continued to one half of its Terms; after which those numbers return in an inverted order to the end of that Series: Which is to be understood in case the number of its Terms should Happen to be even, for if it should Happen to be odd, then that order is to be continued to the greatest half, after which the return is made by omitting the last number. *Fifthly*, that all the numbers of it are Negative.

The Third Series of the first Branch is a part of the second, and expresses the number of Chances there are for *A* to win Two Stakes of *B*, out of the number of Chances there are in the second Series, for *B*, to win Three Stakes of *A*; with this difference, that it begins and ends at an interval from the first and last Terms of the second Series, equal to the number of Stakes which *A* is to win; and that the Terms of it are all Positive.

It

It is to be observed in general that, let the number of these Series be what it will, the Interval between the beginning of the first and the beginning of the second, is to be equal to the number of Stakes which *B* is to win; and that the Interval between the beginning of the second and the beginning of the third, is to be equal to the number of Stakes which *A* is to win; and that these Intervals recur alternately in the same Order. It is to be observed likewise, that all these Series are alternately Positive and Negative.

All the Observations made upon the first Branch of Series belonging also to the second, it would be needless to say any more of them.

Now the sum of all the Series of the first Branch, being added to the sum of all the Series of the second, the aggregate of these sums will be the Numerator of a Fraction expressing the Probability of the Plays terminating in the given number of Games; of which Fraction the Denominator is the Binomial $a+b$ raised to a Power, whose Index is equal to that given number of Games. Thus, supposing that, in the Case of this Problem, both a and b are equal to Unity, the sum of the Series in the first Branch will be 18778, the sum of the Series in the second will be 12393; and the aggregate of both 31171: And the Fifteenth Power of 2 being 32768, it follows, that the Probability of the Plays terminating in Fifteen Games will be $\frac{31171}{32768}$, which being subtracted from Unity, the remainder will be $\frac{1597}{32768}$: From whence we may conclude, that 'tis a Wager of 31171 to 1597, that either *A* in Fifteen Games shall win Two Stakes of *B*, or *B* win Three Stakes of *A*: Which is conformable to what we had before found in our XXXIXth. Problem.

PROBLEM XLI.

TO Find what Probability there is, that in a given number of Games, *A* may be winner of a certain number q of Stakes; and at some other time, *B* may likewise be winner of the number p of Stakes, so that both circumstances may Happen.

SOLU-

SOLUTION.

FIND, by our XLth Problem, the Probability which *A* has of winning, without any Limitation, the number *q* of Stakes: Find also by our XXXIVth Problem the Probability which *A* has of winning that number of Stakes before *B* may Happen to win the number *p*; then from the first Probability subtracting the second, the remainder will express the Probability there is, that both *A* and *B* may be in a circumstance of winning, but *B* before *A*. In the like manner, from the probability which *B* has of winning without any Limitation, subtracting the Probability which he has of winning before *A*, the remainder will express the Probability there is, that both *A* and *B* may be in a circumstance of winning, but *A* before *B*. Wherefore adding these two remainders together, their sum will express the Probability required.

Thus, if it were required to find what Probability there is, that in Ten Games *A* may win Two Stakes, and that at some other time *B* may win Three. The first Series will be found to be,

$$\frac{aa}{a+b} \times 1 + \frac{2ab}{a+b^2} + \frac{5aabb}{a+b^4} + \frac{14a^3b^3}{a+b^6} + \frac{42a^4b^4}{a+b^8}.$$

The second Series will likewise be found to be

$$\frac{aa}{a+b} \times 1 + \frac{2ab}{a+b^2} + \frac{5aabb}{a+b^4} + \frac{13a^3b^3}{a+b^6} + \frac{34a^4b^4}{a+b^8}.$$

The difference of these Series being $\frac{aa}{a+b} \times \frac{a^3b^3}{a+b^6} + \frac{8a^4b^4}{a+b^8}$ expresses the first part of the Probability required, which, in the Case of an equality of Skill between the Gamesters, would be reduced to $\frac{3}{250}$.

The Third Series is as follows,

$$\frac{b^3}{a+b} \times 1 + \frac{3ab}{a+b^2} + \frac{9aabb}{a+b^4} + \frac{28a^3b^3}{a+b^6}.$$

The Fourth Series is

$$\frac{b^3}{a+b} \times 1 + \frac{3ab}{a+b^2} + \frac{8aabb}{a+b^4} + \frac{21a^3b^3}{a+b^6}.$$

The difference of these two Series being $\frac{b^3}{a+b} \times \frac{aabb}{a+b^4} + \frac{7a^3b^3}{a+b^6}$ expresses

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expresses the second part of Probability required, which, in the Case of an equality of Skill, would be reduced to $\frac{11}{512}$. Wherefore the Probability required would in this Case be $\frac{3}{256} + \frac{11}{512} = \frac{17}{512}$. Whence it follows, that 'tis a Wager of 495 to 17, or of 29 to 1 very nearly, that in Ten Games, *A* and *B* may not both be in a circumstance of winning, *viz.* *A* the number *q*, and *B* the number *p* of Stakes. But if, by the conditions of the Problem, it were left indifferent whether *A* or *B* should win the Two Stakes or the Three, then the Probability required would be increased, and become as follows, *viz.*

$$\frac{aa+bb}{a+b)^2} \times \frac{a^3b^3}{a+b)^6} + \frac{8a^4b^4}{a+b)^8}$$

$$- \frac{a^3+b^3}{a+b)^3} \times \frac{aabb}{a+b)^4} + \frac{7a^3b^3}{a+b)^6}.$$

which, in the Case of an equality of Skill between the Gamblers, would be the double of what it was before.

PROBLEM XLII.

TO Find what Probability there is, that in a given number of Games, *A* may win the number *q* of Stakes; with this farther condition, that *B*, during that whole number of Games, may never have been winner of the number *p* of Stakes.

SOLUTION.

FROM the Probability that *A* has to win without any limitation the number *q* of Stakes, subtract the Probability there is that both *A* and *B* may be winners, *viz.* *A* of the number *q*, and *B* of the number *p* of Stakes, and there will remain the Probability required.

But, if the conditions of the Problem were extended to this alternative, *viz.* that either *A* should win the number *q* of Stakes, and *B* be excluded the winning of the number *p*; or that *B* should win the number *p* of Stakes, and *A* be excluded the winning of the number *q*, the Probability that either the one or the other of these two Cases may Happen, will easily be deduced from what we have said.

LEM-

L E M M A I.

IN any Series of Terms, whereof the first Differences are equal, the Third Term will be twice the Second, minus once the First; and the Fourth Term likewise will be twice the Third, minus once the Second: Each following Term being always related in the same manner to the two preceding ones. And as this relation is expressed by the two Numbers $2 - 1$, I therefore call those Numbers the Index of that Relation.

In any Series of Terms, whose second Differences are equal, the Fourth Term will be three times the Third, minus three times the Second, plus once the First: And each Term in such a Series is always related in the same manner to the three next preceding ones, according to the Index $3 - 3 + 1$. Thus, if there be a Series of Squares, such as 4, 16, 36, 64, 100, whose second differences are known to be equal when their Roots have equal Intervals, as they have in this Case, it will be found that the Fourth Term 64 is $= 3 \times 36 - 3 \times 16 + 1 \times 4$, and that the Fifth Term 100 is $= 3 \times 64 - 3 \times 36 + 1 \times 16$. In like manner, if there were a Series of Triangular numbers, such as 3, 10, 21, 36, 55, whose second Differences are known to be equal, when their sides have equal Intervals, as they have in this Case, it will be found that the Fourth Term is $= 3 \times 21 - 3 \times 10 + 1 \times 3$, and that the Fifth Term is $= 3 \times 36 - 3 \times 21 + 1 \times 10$; and so on.

So likewise, if there were a Series of Terms whose Third Differences are equal, or whose Fourth Differences are $= 0$; such as is a Series of Cubes or Pyramidal numbers, or any other Series of numbers generated by the Quantities $ax^3 + bxx + cx + d$, when a, b, c, d being constant Quantities, x is interpreted successively by the Terms of any Arithmetic Progression: Then it will be found that any Term of it is related to the Four next preceding ones, according to the following Index, viz. $4 - 6 + 4 - 1$, whose parts are the Coefficients of the Binomial $a - b$ raised to the fourth Power, the first Coefficient being omitted.

And generally, if there be any Series of Terms whose last Differences are $= 0$. Let the number denoting the rank of that difference be n ; then the Index of the Relation of each Term to as many of the preceding ones as there are Units in n , will be expressed by the Coefficients of the Binomial $a - b$ raised to the Power n , omitting the first.

But

But if the Relation of any Term of a Series to a constant number of preceding Terms, be expressed by any other Indices than those which are comprised under the foregoing general Law; or even if, those Indices remaining, any of their Signs + or — be changed, that Series of Terms will have none of its differences equal to Nothing.

LEMMA II.

IF in any Series, the Terms A, B, C, D, E, F &c. be continually decreasing, and be so related to one another that each of them may have to the same number of preceding Terms a certain given Relation, always expressible by the same Index; I say, that the sum of all the Terms of that Series ad infinitum may always be obtained.

First, Let the Relation of each Term to the two preceding ones be expressed in this manner, viz. Let C be = m Br — n Arr; and let D likewise be = m Cr — n Brr, and so on: Then will the sum of that Infinite Series be equal to $\frac{A+B-mrA}{1-mr+nrr}$.

Thus, if it be proposed to find the sum of the following Series,

A	B	C	D	E	F	G
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 viz. $1r + 3rr + 5r^3 + 7r^4 + 9r^5 + 11r^6 + 13r^7$ &c.
 whose Terms are related to one another in this manner, viz. $C = 2rB - 1rrA$, $D = 2rC - 1rrB$ &c. Let m and n be made respectively equal to 2 and 1, and these Numerical Quantities being Substituted, in the room of the literal ones, in the general Theorem, the sum of the Terms of the foregoing Series will be found to be equal to $\frac{r+3rr-2rr}{1-2r+rr}$, or to $\frac{r+rr}{1-r}$.

Let it be also proposed to find the sum of the following Series

A	B	C	D	E	F	G
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 $1r + 3rr + 4r^3 + 7r^4 + 11r^5 + 18r^6 + 29r^7$ &c.
 whose Terms are related to one another in this manner, viz. $C = 1Br + 1Arr$, $D = 1Cr + 1Brr$ &c. Let m and n be respectively made equal to 1 and — 1, and then that Series will be found equal to $\frac{r+3rr-rr}{1-r-rr}$, or to $\frac{r+2rr}{1-r-rr}$.

DEMON.

DEMONSTRATION.

Let the following Scheme be written down, *viz.*

$$\begin{aligned} A &= A \\ B &= B \\ C &= m Br - n Arr \\ D &= m Cr - n Brr \\ E &= m Dr - n Crr \\ F &= m Er - n Drr \\ &\&c. \end{aligned}$$

This being done, if the sum of the Terms A, B, C, D, E, F &c. *ad infinitum*, composing the first Column, be supposed equal to x , then the sum of the Terms of the other two Columns will be found thus: By *Hypothesis*, $A + B + C + D + E$ &c. $= x$, or $B + C + D + E$ &c. $= x - A$; and Multiplying both sides of this Equation by mr , it will follow that $m Br + m Cr + m Dr + m Er$ &c. is $= mr x - mr A$. Again, adding $A + B$ on both sides, we shall have the sum of the Terms of the second Column, *viz.* $A + B + m Br + m Cr + m Dr$ &c. equal to $A + B + mr x - mr A$. The sum of the Terms of the third Column will be found by bare inspection to be $- n r r x$. But the sum of the Terms contain'd in the first Column, is equal to the other two sums contained in the other two Columns. Wherefore the following Equation will be had, *viz.* $x = A + B + mr x - mr A - n r r x$; from whence it follows that the value of x , or the sum of all the Terms $A + B + C + D + E$ &c. will be equal to

$$\frac{A + B - mr A}{1 - mr + nrr}$$

Secondly, Let the Relation of each Term to the three next preceding ones be expressed as follows, *viz.* let D be $= m Cr - n Brr + p Ar^3$, and let E likewise be $= m Dr - n Crr + p Br^3$, and so on: Then will the sum of all the Terms $A + B + C + D + E$ &c. *ad infinitum*, be equal to

$$\frac{A + B + C - mr A + nrr A - mr B}{1 - mr + nrr - pr^3}$$

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To

To apply this Theorem, let it be proposed to find the sum of the following Series,

A	B	C	D	E	F	G
$4r$	$+ 16rr$	$+ 36r^3$	$+ 64r^4$	$+ 100r^5$	$+ 144r^6$	$+ 196r^7$ &c.

whose Numerical Quantities are related to one another according to the Index $3 - 3 + 1$, corresponding to $m - n + p$: Let therefore 3, 3, 1 be substituted in the room of m, n, p ; let also $4r, -16rr, 36r^3$ be substituted in the room of A, B, C : Then the sum of the Terms of the foregoing Series will be found equal to $\frac{4r + 16rr + 36r^3 - 12rr - 48r^3 + 12r^3}{1 - 3r + 3rr - r^3}$, or $\frac{4r + 4rr}{1 - r^3}$.

And in like manner the sum of the Terms of the following Series, viz.

$r + 2rr + 5r^3 + 20r^4 + 72r^5 + 261r^6 + 947r^7$ &c. whose Numerical Quantities are related to one another according to the Index $3 + 2 + 1$, will by a proper substitution, be found to be equal to $\frac{r - rr - 3r^3}{1 - 3r - 2rr - r^3}$.

Thirdly, Let the Relation of each Term of a Series to four of the preceding Terms be expressed by means of the Index $m - n + p - q$, and the Sum of that Series will be

$$\begin{array}{r} A + B + C + D - mrA + nrrA - pr^3A \\ \quad - mrB + nrrB \\ \quad - mrC \\ \hline 1 - mr + nrr - pr^3 + qr^4 \end{array}$$

Fourthly, The Law of the continuation of these Theorems being manifest, they may be all easily comprehended under one general Rule.

Fifthly, If the corresponding Terms of any two or more Series, generated after the manner which we have above described, be multiplied by one another, the new Series resulting from that multiplication, will also be exactly summable: Thus, taking the two following Series, viz.

$$\begin{array}{l} r + 2rr + 3r^3 + 5r^4 + 8r^5 + 13r^6 \text{ \&c.} \\ r + 3rr + 4r^3 + 7r^4 + 11r^5 + 18r^6 \text{ \&c. in both of} \\ \text{which} \end{array}$$

which each Numerical Quantity is the sum of the two preceding ones; the Series resulting from the multiplication of the corresponding Terms will be

$$rr + 6rr^4 + 12r^6 + 35r^8 + 88r^{10} + 234r^{12} \&c.$$

in which each Numerical Quantity being related to the three preceding ones, according to the Index $2 + 2 - 1$, the sum

$$\text{of that Series will be found to be } = \frac{rr + 4r^4 - 2r^6}{1 - 2rr - 2r^4 + r^6}$$

as will appear, if in the room of $m - n + p$ there be substituted $2 + 2 - 1$, and rr be written instead of r .

When the Numerical Quantities belonging to the Terms of any Series are restrained to have their last differences equal to Nothing; then may the sums of those Series be also found by the following elegant Theorem, which has been communicated to me by *Mr. de Monmort*.

Let Ar be the first Term of the Series, and let the first, second and third differences, &c. of the Numerical Quantities belonging to the Terms of the Series, be respectively equal to d' , d'' , d''' , &c. Then will the sum of the Series be equal to

$$\frac{Ar}{1-r} + \frac{rrd'}{1-r^2} + \frac{r^2d''}{1-r^3} + \frac{r^3d'''}{1-r^4} + \frac{r^4d'''}{1-r^5} \&c.$$

Thus, if it were proposed to find by this Theorem the sum of the following Series, *viz.*

$$Ar + 16rr + 36r^3 + 64r^4 + 100r^5 \&c.$$

It is plain that in this case A is $= 4$, $d' = 12$, $d'' = 8$, $d''' = 0$; and therefore that the sum of this Series is equal to

$$\frac{4r}{1-r} + \frac{12rr}{1-r^2} + \frac{8r^3}{1-r^3}, \text{ which is reduced to } \frac{4r + 4rr}{1-r^3}.$$

R E M A R K.

Our Method of summing up all the Terms which in these Series are related to one another according to constant Indices, may be extended to the finding of the sum of any determinate number of those Terms. Thus, if A, B, C, D be the first Terms of a Series, and U, X, Y, Z be the last, then will the sum of the Series be

$A -$

$$\begin{aligned}
 &A - mr A + nrr A - pr^3 A + qr^4 U \\
 &+ B - mr B + nrr B - pr^3 X + qr^4 X \\
 &+ C - mr C + nrr T - pr^3 T + qr^4 T \\
 &+ D - mr Z + nrr Z - pr^3 Z + qr^4 Z \\
 &\hline
 &1 - mr + nrr - pr^3 + qr^4
 \end{aligned}$$

And if a general Theorem were desired, it might easily be formed from the inspection of the foregoing.

These Theorems are very useful for summing up readily those Series which express the Probability of the Plays being Ended in a given number of Games. For example, suppose it be required to find what Probability there is, that in Four and twenty Games, either *A* shall win Four Stakes of *B*, or *B* Four Stakes of *A*. The Series expressing that Probability is, from our XXXIVth Problem

$$\frac{a^4 + b^4}{a + b^4} \times 1 + \frac{4ab}{a + b^2} + \frac{14aabb}{a + b^4} + \frac{48a^3b^3}{a + b^6} + \frac{164a^4b^4}{a + b^8} \&c.$$

or, supposing an equality of Skill between the two Game-

sters, $\frac{1}{8} \times 1 + \frac{4}{4} + \frac{14}{16} + \frac{48}{64} + \frac{164}{256} \&c.$ which ought to be continued to eleven Terms independently from the common Multiplier. Let this Series, whose Terms are related according to the Index $4 - 2$, be compared with the Theorem, making $A = 1$, $B = \frac{4}{4} = 1$, $m = 4$, $n = 2$. and neglecting the Terms *C*, *D*, *U*, *X*, the sum of the afore-said Series will be found $= 8 + T - 7Z$; which being multiplied by the common Multiplier $\frac{1}{8}$ prefixt to it, the Probability required will be expressed by $1 + \frac{1}{8}T - \frac{7}{8}Z$. Wherefore nothing remains to be done but to find the two last Terms *T* and *Z*: But those two Terms, by our XXXIVth Problem, will be found to be $\frac{76096}{2^{18}}$ and $\frac{259808}{2^{20}}$, or 0.2902, and 0.2477 nearly; which numbers being substituted respectively in the room of *T* and *Z*, the Probability required will be found to be equal to 0.8193 nearly. Let now this last number be subtracted from Unity, and the remainder being 0.1807, it follows, that 'tis a Wager of 82 to 18, or of 41 to 9 nearly, that in Twenty four Games or sooner, either *A* shall win four Stakes of *B*, or *B* four Stakes of *A*.

If

If the number of Stakes were Five, the sum of the Terms of the Series belonging to that Case would also be exprest by means of the two last Terms, supposing any given number of Games, or any proportion of Skill. If the number of Stakes were Six or Seven, the sum of the Series belonging to those Cases would be exprest by means of the three last Terms; If Eight or Nine, by means of the Four last Terms, and so on.

LEMMA III.

IF there be a Series of Numbers, as A, B, C, D, E &c. whose Relation is exprest by any constant Index, and there be another Series of Numbers, as P, Q, R, S, T &c. whose last Differences are equal to Nothing; and each Term of the first Series be Multiplied by each corresponding Term of the second, I say that the Products AP, BQ, CR, DS, ET &c. constitute a Series of Terms, whose Relation may be exprest by a constant Index. Thus if we take the Series 1, 2, 8, 28, 100 &c. whose Terms are related by the Index $3+2$, and each Term of that Series be respectively Multiplied by the corresponding Terms of an Arithmetic Progression, such as 1, 3, 5, 7, 9 &c. whose last Differences are equal to Nothing: Then it will be found that the Products 1, 6, 40, 196, 900 &c. constitute a Series of Numbers, each Term of which is Related to the preceding ones according to the Index $6-5-12-4$. Now the Rule for finding the Index of this Relation is as follows.

Take the Index which expresses the Relation of the Terms in the first Series, and Multiply each Term of it by the corresponding Terms of the Literal Progression r, rr, r^3 &c. which being done, subtract the sum of these Products from Unity; then let the remainder be raised to its Square, if the second Series be composed of Terms in Arithmetic Progression; or to its Cube, if it be composed of Terms whose third Differences are equal to Nothing; or to its fourth Power, if it be composed of Terms whose fourth Differences are equal to Nothing; and so on. Let that Power be subtracted from Unity, and the remainder, having cancelled the Letter r , will be the Index required. Thus in the foregoing Example, having taken the Index $3+2$, which belongs to the first Series, and Multiplied its Terms by r and rr respectively, let the Product $3r + 2rr$

M m be

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be subtracted from Unity, and the Square of the remainder being $1 - 6r + 5rr + 12r^3 + 4r^4$, let that Square be also subtracted from Unity, then the remainder, having cancelled the Letter r , will be $6 - 5 - 12 - 4$, which is the Index required.

But in case neither of the two first Series have any of their last Differences equal to Nothing, yet if in both of them the Relation of their Terms be expressed by constant Indices, the third Series, resulting from the Multiplication of the corresponding Terms of the two first Series, will also have its Terms related to one another according to a constant Index. Thus, taking the Series 1, 3, 5, 11, 21, 43 &c. the Relation of whose Terms is expressed by means of the Index $1+2$, and Multiplying its Terms by the corresponding Terms of the Series 1, 2, 5, 13, 34, 89 &c. the Relation of whose Terms is expressed by the Index $3-1$, the Products will compose the Series 1, 6, 25, 143, 714, 3827, whose Terms are Related to one another according to the Index $3 + 13 - 6 - 4$.

Generally, If the Index expressing the Relation of the Terms in the first Series be $m+n$, and the Index expressing the Relation of the Terms in the second Series be $p+q$; then will the Index of the Relation, in the Series resulting from the Multiplication of the corresponding Terms of the Two first Series, be expressed by the following Quantities,

$$\begin{array}{l} \text{viz.} \quad + mmq \\ mp + npp + mnpq - nnqq. \\ \quad + 2nq \end{array}$$

But if it so Happen that p be equal to m , and q to n ; then the foregoing Theorem may be contracted, and the Index of the Relation may be expressed as follows, viz. $\frac{mm + mmn}{+ n + nn} - n^3$

so that the Relation of each Term to the preceding ones need not be extended, in this Case, to any more than three Terms.

And in like manner other Theorems may be found, which may be extended farther, and at last be comprized under one general Rule.

P R O.

PROBLEM XLIII.

Supposing *A* and *B*, whose proportion of Skill is as *a* to *b*, to Play together, till *A* either wins the number *q* of Stakes, or loses the number *p* of them; and that *B* Sets at every Game the sum *G* to the sum *L*: It is required to find the Advantage, or Disadvantage of *A*.

SOLUTION.

First, Let the number of Stakes to be won or lost on either side be equal, and let that number be *p*; let there be also an equality of Skill between the Gamesters: Then I say, the gain of *A* will be $pp \times \frac{G-L}{2}$, that is, the Square of the number of Stakes which either Gamester is to win or lose, Multiplied by one half of the Difference of the value of the Stakes. Thus, if *A* and *B* play till such time as Ten Stakes are won or lost, and *B* Setts a Guinea to Twenty Shillings; then the gain of *A* will be a hundred times half the Difference between a Guinea and Twenty Shillings, viz. 3*l*—15^{shil.}

Secondly, Let the number of Stakes be unequal, so that *A* be obliged either to win the number *q* of Stakes, or to lose the number *p*; let there be also an equality of Chance between *A* and *B*: Then I say, that the gain of *A* will be $p q \times \frac{G-L}{2}$; that is, the Product of the two numbers of Stakes, and one half of the Difference of the value of the Stakes Multiplied together. Thus, if *A* and *B* play together till such time as either *A* wins Eight Stakes, or loses Twelve; then the Gain of *A* will be the Product of the Three numbers 8, 12, 9, which makes 864 pence, or 3*l*—12^{shil.}

Thirdly, Let the number of Stakes be equal, but let the number of Chances to win a Game, or the Skill of the Gamesters be unequal, in the proportion of *a* to *b*. Then I say,

that the gain of *A* will be $\frac{p a^p - p b^p}{a^p + b^p} \times \frac{a G - b L}{a - b}$.

Fourthly,

Fourthly, Let the number of Stakes be unequal, and let also the number of Chances be unequal: Then I say that the

gain of A will be $\frac{q a^q \times a^p - b^p - p b^p \times a^q - b^q}{a^{p+q} - b^{p+q}} \times \frac{a G - b L}{a - b}$

DEMONSTRATION.

IN order to form a general Demonstration of these Rules, let us resolve some particular Cases of this Problem, and examine the process of their Solution: Let it therefore be proposed to find the gain of A in the Case of Four Stakes to be won or lost on either side, and of an equality of Chance between A and B to win a Game. There being an equality of Chance for A , every Game he plays, to win G or to lose L , it follows, that the gain of every Game he plays is to be reputed to be $\frac{G-L}{2}$. But it being uncertain whether any more Games than Four will be play'd, it follows, that the gain of the Tenth Game, for instance, to be estimated before the play begins, cannot be reputed to be $\frac{G-L}{2}$; for it would only be such provided the Play were not Ended before that Tenth Game: Wherefore the gain of the Tenth Game is the Quantity $\frac{G-L}{2}$ Multiplied by the Probability of the Plays not being Ended in Nine Games, or before, for the same reason, the gain of the Ninth Game is the Quantity $\frac{G-L}{2}$ Multiplied by the Probability that the Play will not be Ended in Eight Games: And likewise the gain of the Eighth Game is the Quantity $\frac{G-L}{2}$ Multiplied by the Probability that the Play will not be ended in Seven Games, and so on. From whence it may be concluded, that the gain of A , to be estimated before the Play begins, is the Quantity $\frac{G-L}{2}$ Multiplied by the sum of the Probabilities that the Play will not be Ended in 0, 1, 2, 3, 4, 5, 6, &c. Games *ad infinitum*.

Let those Probabilities be respectively called $A', B', C', D', E', F', G',$ &c. Then, because the Probability of the Plays not being Ended in Five Games is equal to the Probability of its not Ending in Four, and that the Probability of its not Ending in Seven, is equal to the Probability of its not Ending

Ending in Six, it will follow, that the sum of the Probabilities belonging to all the Even Games is equal to the sum of the Probabilities belonging to all the Odd ones: We are therefore only to find the sum of all the Even Terms, $A' + C' + E' + G' \&c.$ and to double it afterwards.

Now it will appear, from our XXXIII^d Problem, that these Terms constitute the following Series, *viz.*

$$\frac{1}{1} + \frac{4}{4} + \frac{14}{16} + \frac{48}{64} + \frac{164}{256} + \frac{460}{1024} + \frac{1912}{4096} \&c.$$

In which Series, each Numerator being Related to the two preceding ones according to the Index $4 - 2$, and each Denominator being a Power of 4, it follows, that this Series may be compared with the first Theorem of our second Lemma; by making the first and second Terms A and B , used in that Theorem, to be respectively equal to 1 and $\frac{4}{4}$, making also the Quantities m, n, r respectively equal to 4, 2, $\frac{1}{4}$. Which being done, it will be found that the sum of all those Terms *ad infinitum* will be equal to 8.

We may therefore conclude that the sum of all the Terms $A' + B' + C' + D' + E' \&c.$ is equal to 16, and that the gain of A is equal to $16 \times \frac{6-1}{2}$.

But if the number of Chances which A and B have to win a Game, be in a proportion of inequality, then the sum of the Series $A' + C' + E' + G' + I' \&c.$ will be found thus: Let $\frac{ab}{a+b^2}$ be called r , and the Terms of that Series will be Related to one another as follows, *viz.* $E' = 4C'r - 2A'rr$, $G' = 4E'r - 2C'rr$, and so on. Let therefore 4, 2, 1, 1, be respectively substituted, in the first Theorem of our second Lemma, in the room of m, n, A', B' ; and the sum of this Series will be found to be $\frac{2-4r}{1-4r+2rr}$; in which ex-

pression, restoring the value of r , *viz.* $\frac{ab}{a+b^2}$, the sum of the Series will become $\frac{2aa+2bb \times a+b^2}{a^2+b^2}$, the double of which is the sum of all the Terms $A' + B' + C' + D' + E' \&c.$ But because, in every Game, the Gamester A has the number a of Chances to win G , and the number b of Chances to lose L ; it follows, that his gain in every Game is equal

to $\frac{aG - bL}{a + b}$. From whence it may be concluded, that the Advantage of A , to be estimated before the Play begins,

$$\text{will be } \frac{aaa + 4bb \times a + b}{a^4 + b^4} \times \frac{aG - bL}{a + b}.$$

Before we proceed farther, we must observe, that the Series $A' + B' + C' + D' + E' + F' \&c.$ which we have assumed to represent in general the Probabilities of the Plays not being Ended in 0, 1, 2, 3, 4, 5 &c. Games, whether the Stakes be equal or unequal, being divided into two parts, *viz.* $A' + C' + E' + G' \&c.$ and $B' + D' + F' + H' \&c.$ answering to 0, 2, 4, 6 &c. and 1, 3, 5, 7, &c. each Term of these two new Series will be related to the preceding ones, according to the same Law of Relation, as are the Terms of those Series which express the Probabilities of the Plays being Ended in a certain number of Games, under the like circumstances of Stakes to be won or Lost. The Law of which Relation is to be deduced from our XXXIVth, and XXXIXth Problems.

If the number of Stakes to be won or lost on either side be equal to Six, and the proportion of Chances to win a single Game be as a to b ; then the Relation of each Term to the preceding ones, in the Series $A' + C' + E' + G' \&c.$ will be expressed by the Index $6 - 9 + 2$. Wherefore to find the sum of these Terms, let the Quantities 6, 9, 2, 1, 1, 1, be respectively substituted, in the third Theorem of our second Lemma, in the room of m, n, p, A', B', C' , and the sum of those Terms will be found to be

$$\frac{1 + 1 + 1 - 6r + 9rr}{1 - 6r + 9rr - 2r^3} \text{ or } \frac{3 - 12r + 9rr}{1 - 6r + 9rr - 2r^3}. \text{ In which ex-}$$

pression substituting $\frac{ab}{a+b}$ in the room of r , the same will

$$\text{become } \frac{3a^4 + 3aabb + 3b^4 \times a + b}{a^6 + b^6}. \text{ From whence we may}$$

conclude, that the gain of A will be

$$\frac{3a^4 + 3aabb + 3b^4 \times a + b}{a^6 + b^6} \times \frac{aG - bL}{a + b}.$$

Again,

Again, if the number of Stakes to be won or lost on either side be Eight, it will be found, that the gain of A will

$$\text{be } \frac{8a^6 + 8a^4bb + 8a^2b^2b + 8b^4 \times a + b^2}{a^8 + b^8} \times \frac{aG - bL}{a + b}.$$

But the Numerators of the foregoing Fractions being in Geometric Progression, if those Progressions be summed up, the gain of A , in the Case of Four Stakes to be won or lost, may be expressed as follows,

$$\text{viz. by the Fraction } \frac{4a^4 - 4b^4 \times a + b^2}{a^4 + b^4 \times aa - bb} \times \frac{aG - bL}{a + b}; \text{ or}$$

dividing both Numerator and Denominator by $a + b^2$, the same may be expressed by the Fraction $\frac{4a^2 - 4b^2}{a^2 + b^2} \times \frac{aG - bL}{a - b}$.

His gain likewise, in the Case of Six Stakes to be won or lost, will be expressed by the Fraction $\frac{6a^6 - 6b^6}{a^6 + b^6} \times \frac{aG - bL}{a - b}$; and

in the Case of Eight Stakes to be won or lost, it will be expressed by the Fraction $\frac{8a^8 - 8b^8}{a^8 + b^8} \times \frac{aG - bL}{a - b}$: So that we

may conclude, that in any Case of an Even and equal number of Stakes denominated by p , the gain of A will be expressed

$$\text{by the Fraction } \frac{pa^p - pb^p}{a^p + b^p} \times \frac{aG - bL}{a - b}.$$

But if the number of Stakes be Odd and equal, as it is in the Case of Five Stakes to be won or lost, then the two Series $A' + C' + E' + G' + I'$ &c. and $B' + D' + H'$ &c. will be unequal, and the excess of the first above the second will be Unity. Wherefore to find the gain of A , in the Case of Five Stakes, having set aside the first Term of the first Series, let all other the Terms be added together, by comparing them with those that are employed in the first Theorem of our second Lemma; which will be done thus. Since $C' = 1$, $E' = 1$, and $G' = 5E'r - 5C'rr$, let the numbers 1, 1, 5, 5, be respectively substituted in the aforesaid Theorem, in the room of the Letters A' , B' , m , n ; and

and the sum of that Series will be found to be $\frac{2 - 5r}{1 - 5r + 5rr}$.

To the double of which adding Unity, which we had set aside, it will appear that the sum of the two Series together

will be $\frac{5 - 5r + 5rr}{1 - 5r + 5rr}$; or writing $\frac{ab}{a+b}$ in the room of r ,

$\frac{5a^4 + 5a^3b + 5aabb + 5ab^3 + 5b^4}{a^4 - a^3b + aabb - ab^3 + b^4}$. Now by reason that the

Terms of both Numerator and Denominator of this last Fraction compose a Geometrick Progression, the Numerator

will be reduced to $\frac{5a^5 - 5b^5}{a - b}$, and the Denominator will

be reduced to $\frac{a^5 + b^5}{a + b}$. From whence it follows, that the

sum of these two Series will be $\frac{5a^5 - 5b^5 \times a + b}{a^5 + b^5 \times a - b}$, and that

the gain of A will be $\frac{5a^5 - 5b^5}{a^5 + b^5} \times \frac{aG - bL}{a - b}$. If the

gain of A be likewise inquired into, in the Case of Seven Stakes to be won or lost, then it will be found to be

$\frac{7a^7 - 7b^7}{a^7 + b^7} \times \frac{aG - bL}{a - b}$. And the same form of expression

being constantly observed in all cases wherein the number of Stakes is Odd and equal, we may conclude that if that number be denominated by p , then the gain of A will be

$\frac{p a^p - p b^p}{a^p + b^p} \times \frac{aG - bL}{a - b}$. Now this expression of the

gain of A having been found to be the same in the Case of an Even number of Stakes, as it is now found in the Case of an Odd one; we may conclude, that it is general, and belongs to any equal number of Stakes whether Even or Odd.

If the number of Stakes be unequal, the Investigation of the gain of A will be made in the same manner as it was in the Case of an equality of Stakes. Thus, let us suppose that the Play be to continue till such time as either A wins Two Stakes, or B Three. In order therefore to find the gain of A , let the Series $A' + B' + C' + D' + E' + F' \&c.$ be

be divided into two parts, viz. $A' + C' + E'$ &c. and $B' + D' + F'$ &c. then it will appear, from our XXXIII^d, and XXXIXth Problems, that $A' = 1$, $C' = \frac{2ab+bb}{a+b}$,

$E' = 3C'r - 1A'rr$. Having now obtained the first Terms of the Series, and the Relation of each Term of it to the preceding ones; it will be easie to find the sum of all its Terms, by the help of the first Theorem of our second Lemma, making the Quantities, A, B, m, n therein employed to be respectively equal to 1, $\frac{2ab+bb}{a+b}$, 3, 1. This done, the sum of that Series will be found to be equal to

$$\frac{aa+ab+2bb \times \overline{a+b}}{a^4+a^3b+aaab+ab^3+b^4}. \text{ In like manner it will appear,}$$

$$\text{that in the second Series } B' \text{ is } = 1, D' = \frac{2aab+3abb}{a+b},$$

$$F' = 3D'r - B'rr; \text{ from whence the sum of all its Terms will be found to be } \frac{aa+2ab+2bb \times \overline{a+b}}{a^4+a^3b+aaab+ab^3+b^4}; \text{ And both}$$

$$\text{sums of those Series being added together, the aggregate of them will be } \frac{2a^3+aab+6abb+3b^3 \times \overline{a+b}}{a^4+a^3b+aaab+ab^3+b^4}; \text{ But the}$$

Terms of this Denominator composing a Geometric Progression, whose sum is $\frac{a^5-b^5}{a-b}$, the foregoing Fraction may

$$\text{be reduced to } \frac{2a^4+2a^3b+2aaab-3ab^3-3b^4 \times \overline{a+b}}{a^5-b^5}; \text{ which}$$

Fraction is still capable of a farther reduction; for the three first Terms of its Numerator compose a Geometric Progression, and the two last Terms may be considered as being in Geometric Progression, and consequently the Fraction may

$$\text{at last be reduced to } \frac{2aa \times a^3 - b^3 - 3b^3 \times aa - bb \times \overline{a+b}}{a^5-b^5 \times a-b},$$

from which expression, the gain of A will be found to be

$$\frac{2aa \times a^3 - b^3 - 3b^3 \times aa - bb}{a^5-b^5} \times \frac{aG-bL}{a-b}.$$

O o

By

By the same method of Process, it will be easy to determine the gain of *A* under any other circumstance of Stakes to be won or lost: And if it be remembred always to sum up those Terms which are in Geometric Progression, all the various expressions of the gain of *A*, calculated for differing numbers of Stakes, will appear to be uniform: From whence it may be collected by bare inspection, that the gain of *A* is what we have asserted it to be, viz.

$$\frac{qaq \times ap - bp - pbp \times aq - bq}{ap+q - bp+q} \times \frac{aG - bL}{a-b}.$$

It is to be observed, *First*, that if *p* and *q* be equal, the foregoing expression may be reduced to $\frac{pap - pbp}{ap + bp} \times \frac{aG - bL}{a-b}$, as will appear if both Numerator and Denominator be divided by $a^p - b^p$, having first substituted *p* in the room of *q*. *Secondly*, that if *a* and *b* be equal, the same expression may be reduced to $p q \times \frac{G-L}{2}$, which will appear if both Numerator and Denominator be divided by $a-b$.

After I had Solved the foregoing Problem, I wrote word of it to Mr. *Nicolas Bernoulli*, the present Professour of Mathematics at *Padoua*, without acquainting him with my Solution: I only let him know in general that it was done by the Method of Infinite Series; whereupon he sent me two different Solutions of that Problem: And as one of them has some Affinity with the Method of Series used all along in this Book, I shall transcribe it here in the Words of his Letter, "My Uncle has observed that this Problem may also be Solved after the same manner as you have Solved the "Ninth Problem" of your *Tract de Mensura Sortis*, it being visible that the Expectations of the Gamesters will receive no alteration whether it be supposed that the Pieces which *A* and *B* Set every time to each other, are respectively *L* and *G*, or whether it be supposed that those Pieces constitute the following Progression, viz.

" $L, G, G + \frac{b}{a} \times \overline{G-L}, G + \frac{b}{a} \times \frac{bb}{aa} \times \overline{G-L}, G + \frac{b}{a} \times \frac{bbb}{aaa} \times \overline{G-L}$ &c. the number of whose Terms is $p+q$, whereof

* See the 1Xth Problem of this Book,

“ the first, whose number is p , denote the Pieces of A ; and
 “ the last, whose number is q , denote the Pieces of B : For in
 “ either Case the gain of A will be $\frac{aG-bL}{a+b}$. Now it being
 “ possible to find the sum of any number of Terms of this Pro-
 “ gression, it follows that the different values of all the Pieces
 “ of each Gamester may be obtained: Let therefore those
 “ values be denoted respectively by S and T ; let also the
 “ Probabilities of winning the number of Stakes agreed up-
 “ on be called A and B respectively, which Probabilities are
 “ $\frac{a^p b^q - a^q b^p}{a^p b^q - b^p a^q}$ and $\frac{a^q b^p - b^p a^q}{a^p b^q - b^p a^q}$, such as we had severally
 “ derived them, your self in your aforesaid Problem, and I
 “ in Mr. *Montmort*’s Book. This being supposed, the gain of A
 “ will be found to be $AT - BS$, or $\frac{aG-bL}{a+b} \times \frac{Aq-Bp}{a+b}$.

N. B. Tho’ I may, accidentally, have given a useful Hint
 for that elegant Method of solving the foregoing Problem, yet
 I think it reasonable to ascribe it entirely to its proper Author;
 the Hint having been improv’d much beyond what I could
 have expected.

REMARK.

IT is to be observed, that the gain of A is not to be regu-
 lated by the equal Probability there is that the Play may,
 or may not be Ended in a certain number of Games. For
 instance, If two Gamesters having the same number of Chan-
 ces to win a Game, design only to play untill such time only
 as two Stakes are won or lost; it is as Probable that the Play
 may be Ended in two Games as not, yet it cannot be con-
 cluded from thence, that the gain of A is to be estimated
 by the Product of the Number 2 by one half of the Diffe-
 rence of the Stakes: For it has been Demonstrated that this
 gain will be Four times that half difference. In like manner,
 if the Play were to continue, till either A should win Two
 Stakes, or B Three; it will be found, that it is as Probable
 that the Play may End in Four Games as not; and yet the
 gain of A is not to be estimated by the Product of the Num-
 ber 4 by one half of the Difference of the Stakes; it ha-
 ving

ving been Demonstrated that it is Six times that half Difference. To make this the more sensible, let us suppose that *A* and *B* are to Play till such time as *A* either wins one Stake, or loses Ten: It is plain, that in this Case it is as Probable that the Play may be Ended in One Game as not, and yet the gain of *A* will be found to be Ten times the Difference of the Stakes. From hence it is plain, that this gain is not to be estimated, by the equal Probability of the Plays Ending or not Ending in a certain number of Games, but by the Rules which have been prescribed in this Problem.

PROBLEM XLIV.

IF *A* and *B*, whose proportion of Skill is as *a* to *b*, resolving to Play together till such time as Four Stakes are won or lost on either side, agree between themselves, that the first Game that is play'd, they shall Set to each other the respective sums *L* and *G*; that the second Game they shall Set the sums $2L$ and $2G$; the third Game the sums $3L$ and $3G$, and so on; the Stakes increasing continually in an Arithmetic Progression: It is Demanded how the gain of *A* is to be estimated in this Case, before the Play begins.

SOLUTION.

LET there be supposed a Time wherein the Number *p* of Games has been play'd; then *A* having the Number *a* of Chances to win the sum $p+1 \times G$ in the next Game, and *B* having the Number *b* of Chances to win the sum $p+1 \times L$; it is plain, that the gain of *A* in that circumstance of Time will be $p+1 \times \frac{aG-bL}{a+b}$. But this gain being to be estimated before the Play begins, it follows, that it ought to be estimated by the Quantity $p+1 \times \frac{aG-bL}{a+b}$ multiplied by the respective Probability there is that the Play will not then be Ended; and therefore the whole gain of *A* is the sum of the Probabilities of the Plays not Ending in 0, 1, 2, 3, 4, 5, 6 &c. Games *ad infinitum*, multiplied by the respective values of the Quantity $p+1 \times \frac{aG-bL}{a+b}$, *p* being Interpreted

preted successively by the Terms of the Arithmetic Progression, 0, 1, 2, 3, 4, 5, 6 &c. Now let these Probabilities of the Plays not Ending be respectively called A' , B' , C' , D' , E' , F' , G' &c. Let also the Quantity $\frac{aG-bL}{a+b}$ be called S ; and thence it will follow, that the gain of A will be $A'S + 2B'S + 3C'S + 4D'S + 5E'S + 6F'S$ &c. But in the Case of this Problem B' is equal to A' , and D' is equal to C' , and so on. Wherefore the gain of A may be expressed by the Series $S \times 3A' + 7C' + 11E' + 15G' + 19I$ &c. But it appears, by our XXXIII^d Problem, that the Terms A' , C' , E' , G' are respectively equal to the following

Quantities, viz. 1, 1, $\frac{4a^3b + 6aabb + 4ab^3}{a+b^4}$, $\frac{14a^4bb + 20a^3b^3 + 14aabb^4}{a+b^6}$: Whence it follows, that

the Terms $3A' + 7C' + 11E' + 15G'$ may be obtained: It appears also, from what we have observed in the preceding Problem, that the Relation of the Terms A' , C' , E' &c. may be expressed by the Index $4 - 2$; and by the Third Lemma prefix to that Problem; that the Relation of the Terms $3A'$, $7C'$, $11E'$ &c. may be expressed by the Index $8 - 20 + 16 - 4$: And therefore substituting the Quantities $3A'$, $7C'$, $11E'$, $15G'$ in the room of the Quantities A, B, C, D , which we make use of in the Third Theorem of our second Lemma; substituting likewise the Quantities, 8, 20, 16, 4 in the room of m, n, p, q ; and lastly substituting $\frac{ab}{a+b^2}$ in the room of r ; the gain of A will be exprest by the following Quantities, viz.

$$S \times \frac{10a^6 + 24a^5b + 42a^4bb + 64a^3b^3 + 42aabb^4 + 24ab^5 + 10b^6}{(a^4 + b^4)^2} \times \frac{a+b^2}{a+b}$$

which, in the Case of an equal number of Chances to win a Stake, would be reduced to $216S$; and therefore if the Quantities G and L stand respectively for a Guinea and Twenty Shillings, which will make the value of S to be Nine pence, it follows, that the gain of A will in this Case be $8\text{ l} - 2\text{ shil.}$

Corollary I. If the Stakes were to Increase according to the proportion of the Terms of any of those Series which we have described in our *Lemma's*, and that there were any given inequality in the number of Stakes to be won or lost, the gain of *A* might still be found.

Corollary II. There are some Cases wherein the gain of *A* would be Infinite: Thus, if *A* and *B* were to Play till such times as Four Stakes were won or lost, and it were agreed between them to double their Stakes at every Game, the gain of *A* would in this Case be Infinite: Which consequence may easily be deduced from what has been said before.

PROBLEM XLV.

IF *A* and *B* resolve to Play till such time as *A* either wins a certain given number of Stakes, or that *B* wins the same, or some other given number of them: 'Tis required to find in how many Games it will be as Probable that the Play may be Ended as not?

SOLUTION.

LET it be supposed that *A* and *B* are to play till such time as either of them wins Three Stakes, and that there is an Equality of Skill between them. This being supposed, it will appear, from our XXXIVth Problem, that the Probability of the Plays continuing for an Indeterminate number of Games may be exprest by the following Series, viz.

$$\frac{a^3 + b^3}{a + b^3} \times 1 + \frac{3ab}{a + b^2} + \frac{9aabb}{a + b^4} + \frac{27a^3b^3}{a + b^6} \&c. \text{ which, in}$$

the Case of an Equality of Skill between the Gamesters, will be reduced to this Series,

$$\frac{2}{8} \times 1 + \frac{\text{III}}{4} + \frac{\text{V}}{16} + \frac{\text{VII}}{64} + \frac{\text{IX}}{27} \&c. \text{ whose Terms are respective-}$$

ly corresponding to the number of Games 3, 5, 7, 9 &c. Wherefore so many of those Terms ought to be taken, as that their sum being multiplied by the common Multiplicator

for $\frac{2}{8}$ or $\frac{1}{4}$, the Product may be equal to the Fraction $\frac{1}{2}$, which Fraction denotes the equal Probability of an Events Happening or not Happening: But if two of those Terms be taken, and that their sum be Multiplied by $\frac{1}{4}$, the Product will be $\frac{7}{16}$; which being less than the Fraction $\frac{1}{2}$, it may be concluded that Five Games are too few to make it as Probable that the Play will be Ended in that number of Games as not; and that the Odds against its Ending in Five Games are 9 to 7. But if Three of those Terms be taken, then their sum being multiplied by the common Multiplicator $\frac{1}{4}$, the Product will be $\frac{37}{64}$; which exceeding the Fraction $\frac{1}{2}$, it may be concluded that Seven Games are too many; and that the Odds of the Play being Ended in Seven Games, or sooner, are 37 to 27, or 4 to 3 very nearly.

N. B. It would be needless to inquire whether Six Games might not bring the Play to an equal Probability of Ending or not Ending; it having been observed before, that in the Case of an equality of Stakes to be play'd for, it is impossible that the Play should End in an Even number of Games, if the number of Stakes be Odd; or that it should End in an Odd number of Games, if the number of Stakes be Even.

In like manner, if the Play were to continue till Four Stakes be won or lost on either side: Then taking the following Series, viz.

$$\frac{a^4 + b^4}{a + b^4} \times 1 + \frac{4ab}{a + b^4} + \frac{14aabb}{a + b^4} + \frac{48a^3b^3}{a + b^4} \&c. \text{ which, upon}$$

the supposition of an equality of Skill between the Gamesters, may be reduced to this, viz.

$$\frac{2}{16} \times 1 + \frac{4}{4} + \frac{14}{16} + \frac{48}{64} + \frac{164}{256} \&c. \text{ let so many}$$

of its Terms be tried, as will make the Product of their sum multiplied by $\frac{2}{16}$, equal to the Fraction $\frac{1}{2}$, or as near it as possible. Now Five of those Terms being tried, and their sum being multiplied by $\frac{2}{16}$, or $\frac{1}{8}$, the Product will be $\frac{1092}{2048}$, which not differing much from $\frac{1}{2}$, it may be con-

concluded that Twelve will be very near that number of Games, which will make the Probabilities of the Plays Ending or not Ending to be equal; the Odds for its Ending being only 1092 to 956, or 8 to 7 very nearly. But the Odds against its Ending in Ten Games, will be found to be 39 to 29, or 4 to 3 nearly.

By the same method of Process, it will be found that Five Stakes will probably be won or lost in about Seventeen Games: It being but the Odds of 11 to 10 nearly, that the Play will not be Ended in that number of Games, and 10 to 9 nearly, that it will be Ended in Nineteen.

It will also be found that Six Stakes will probably be won or lost in about Twenty Six Games, there being but the Odds of 168 to 167 nearly, that the Play will not be Ended in that number of Games, and 25 to 22 nearly, that it will be Ended in Twenty Eight.

If the same Method of Trial be applied to any other number of Stakes, whether equal or unequal, and to any proportion of Skill, the number of Games required will always be found.

Yet if the number of Stakes were great, those Trials would become tedious, notwithstanding the Help that might be derived from our Second *Lemma*, whereby any number of Terms of those Series which are employed in the Solution of this Problem, may be added together. For which reason it will be convenient to make some Trials of another nature, and to see whether, from the resolution of some of the simplest Cases of this Problem, any Analogy can be observed between the number of Stakes given, and the number of Games which determine the equal Probability of the Plays Ending or not Ending.

Now Mr de Moivre having with great Sagacity discovered that Analogy, in the Case of an equal and Odd number of Stakes, on supposition of an equality of Skill between the Gamesters, I thought the Reader would be well pleased to be acquainted with the Rule which he has given for that purpose, and which is as follows.

Let n be any Odd number of Stakes to be won or lost on either side; let also $\frac{n+1}{2}$ be made equal to p : Then the Quantity $3pp - 3p + 1$ will denote a number of Games, wherein

wherein it will be more than an equal Chance that the Play will be Ended; thus, if the number of Stakes be Nineteen, then p will be 10, and the Quantity $3pp - 3p + 1$ will be 271, which shews that 'tis more than an equal Chance that the Play will be Ended in 271 Games.

The Author of this Rule owns that he has not been able to find another like it, for an Even number of Stakes; but I am of opinion, that tho' the same Rule, being applied to that Case, may not find the just number of Games wherein there will be more than an equal Probability of the Plays Ending, yet it will always find a number of Games, wherein it is very near an equal Wager that the Play will be Ended. Wherefore to make the Rule as extensive as it may be, I would Chuse to express it by the number of Stakes whether Even or Odd, and make it $\frac{3}{4}nn$, which differs from his own, but by the small Fraction $\frac{1}{4}$.

If any one has a mind to carry this speculation still farther, and to try whether some general Rule may not be discovered for determining, by a very near approximation, the number of Games requisite to make it a Wager of any given proportion of Odds, that the Play will be Ended in that number of Games, whether the Skill of the Gamesters be equal or unequal; let him Solve several Cases of this Problem in the following manner, which I take to be as expeditious as the nature of the Problem can admit of.

Upon a Diameter equal to Unity, if so be the Skill of the Gamesters be equal; or to the Quantity $\frac{4ab}{a+b^2}$, if their Skill be in the proportion of a to b , let a Semicircle be described, which divide into so many equal parts as there are Stakes to be won or lost on either side, supposing those Stakes to be equal. From the First, Third, Fifth, Seventh &c. Points of Division, beginning from one extremity of the Diameter, let Perpendiculars fall upon that Diameter, which by their concurrence with it, shall determine the *versed Sines* of so many Arcs, to be taken from the other extremity thereof. Let the greatest of those *versed Sines* be called m , the next less p , the next to it q , the next s &c. Make also

Q q

$\frac{1}{1-p}$

$$\frac{1-p \times 1-q \times 1-s}{m-p \times m-q \times m-s} \&c. = A$$

$$\frac{1-q \times 1-s \times 1-m}{p-q \times p-s \times p-m} \&c. = B$$

$$\frac{1-s \times 1-m \times 1-p}{q-s \times q-m \times q-p} \&c. = C$$

$$\frac{1-m \times 1-p \times 1-q}{s-m \times s-p \times s-q} \&c. = D$$

&c.

then will the Probability of the Play's not Ending in a number of Games denominated by x , be exprest by the Quantities

$m^{\frac{1}{2}x} A + p^{\frac{1}{2}x} B + q^{\frac{1}{2}x} C + s^{\frac{1}{2}x} D \&c.$ if the number of Stakes be Even, or by the Quantities

$m^{\frac{x-1}{2}} A + p^{\frac{x-1}{2}} B + q^{\frac{x-1}{2}} C + s^{\frac{x-1}{2}} D \&c.$ if the number of Stakes be Odd.

EXAMPLE I.

LET it be required to find what Odds there is, that in 40 Games there will be Four Stakes won or lost on either side.

Having divided the Semicircle into Four equal parts, according to the abovementioned directions, the Quantity m will be the *Versed Sine* of 135 Degrees, and the Quantity p will be the *Versed Sine* of 45 Degrees, which by the help of a Table of *Sines* will readily be found to be 0.85355 and 0.14645 respectively. Moreover the Quantity A being equal to $\frac{1-p}{m-p}$, and the Quantity B to $\frac{1-q}{p-q}$, will be found to be 1.2071 and - 0.2071. From whence it follows, that the Probability of the Plays not Ending in Forty Games may be expres'd by the two following Products $0.85355^{40} \times 1.2071$ — $0.14645^{40} \times 0.2071$, of which the Second may be entirely neglected, as being inconsiderably little in respect of the

the first. Now the Logarithm of the first Product being 2.7063225 , to which answers the number 0.05085 , let that number be subtracted from Unity; and the remainder being 0.94915 , I conclude that the Odds of the Plays Ending in Forty Games are as 94915 to 5085 , or very near as 19 to 1 .

EXAMPLE II.

LET it be required to find how many Games must be play'd, to make it a Wager of 100 to 1 , that Four Stakes will be won or lost on either side, in that number of Games.

Let x be the number of Games required: Then by the foregoing Example it will appear that we may have the Equation $\overline{0.85355}^{\frac{1}{x}} \times 1.2071 = \frac{1}{100}$, in which the value of x may easily be obtained by Logarithms; it being found by one single Division to be about 60 .

If the Stakes be unequal, the Solution will consist of two Series, in both which the Quantities m, p, q &c. will be of the same value, and will be determined likewise by a Table of *Sines*. In this Case the Semicircumference ought to be divided into as many equal parts as there are Units in the number of all the Stakes: Thus, if the Stakes were Four and Five, the Semicircumference ought to be divided into Nine equal parts: But then it is to be observed that the *versed Sines* of those Arcs, which, in the Case of Nine Stakes for each Gamester, are alternately omitted, are those which, in the Case of Four and Five, are to represent the Quantities m, p, q &c. It is to be observed also that the Quantities A, B, C, D &c. by which the Terms of the first Series are to be respectively Multiplied, will be found to differ from the Quantities A', B', C', D' &c. by which the Terms of the second Series are also to be respectively Multiplied; and that both those Series of Quantities may be determined by proper Theorems contrived for that purpose.

Before I make an End of this Subject, I shall propose an Inquiry to be made by those who have sufficient leisure to Try the foregoing Methods; which is, whether the number of Games, wherein it will be an equal Wager that the Play will

will be Ended, upon the supposition of an equal number n of Stakes to be won or lost on either side; as also of the proportion of Skill exprest by a and b , may not be determined very nearly by the following Expreffion, *viz.*

$$\frac{na^n - nb^n}{a^n + b^n} \times \frac{aa + ab + bb}{aa - bb}.$$

PROBLEM XLVI.

IF A and B, whose proportion of Skill is supposed equal, play together till Four Stakes be won or lost on either side; and that C and D, whose proportion of Skill is also supposed equal, play likewise together till Five Stakes be won or lost on either side: What is the Probability that the Play between A and B will be Ended in fewer Games than the Play between C and D?

SOLUTION.

THE Probability of the First Play's being Ended in any number of Games before the Second, is compounded of the Probability of the First Play's being Ended in that number of Games, and of the Second's not being Ended with the Game immediately preceding: From whence it follows, that the Probability of the First Plays Ending in an Indeterminate number of Games before the Second, is the sum of all the Probabilities *ad Infinitum* of the First Play's Ending, Multiplied by the respective Probabilities of the Second's not being Ended with the Game immediately preceding.

But it appears from our XXXIVth Problem, that the Probability of the first Play's Ending in an Indeterminate number of Games, may be exprest by the following Series, *viz.*

$$\begin{array}{cccccc} \text{IV} & \text{VI} & \text{VIII} & \text{X} & \text{XII} & \text{XIV} \\ \frac{1}{2^3} & + \frac{4}{2^5} & + \frac{14}{2^7} & + \frac{48}{2^9} & + \frac{164}{2^{11}} & + \frac{560}{2^{13}} & \&c. \end{array}$$

It appears also, from our XXXIII^d Problem, that the Probability of the Second Play's not Ending may be exprest by the following Series, *viz.*

$$\begin{array}{cccccc} \text{III} & \text{V} & \text{VII} & \text{IX} & \text{XI} & \text{XIII} \\ \frac{4}{2^3} & + \frac{15}{2^5} & + \frac{55}{2^7} & + \frac{200}{2^9} & + \frac{725}{2^{11}} & + \frac{2625}{2^{13}} & \&c. \end{array}$$

Now

Now the Corresponding Terms of those two Series being Multiplied together, the Products, supposing r equal to the Fraction $\frac{1}{16}$, will compose the following Series, viz.

$2r + 30rr + 385r^3 + 4800r^4 + 59400r^5$ &c. in which Series the Index of the Relation of each Numerical Quantity to the preceding ones, may be found by the help of our Third Lemma: For the Index of the Relation in the Numerator of the First Series being $4 - 2$, and the Index of the Relation in the Numerator of the Second being $5 - 5$, which Relations are deduced from the XXXIVth Problem, it follows, that if in the Theorem of our Third Lemma, the Quantities $4, -2, 5, -5$, be respectively substituted in the room of the Quantities m, n, p, q , the Index of the Relation in the Third Series will be found to be $20 - 110 + 200 - 100$; wherefore all the Terms of this Series may be summed up by the Third Theorem of our Second Lemma, substituting the Quantities $20, 110, 200, 100$ in the room of the Quantities m, n, p, q , therein employed; substituting also the Terms $2r, 30rr, 385r^3, 4800r^4$ in the room of the Quantities A, B, C, D : For after those Substitutions, the sum of the Third

Series will be found to be $\frac{2r - 10rr + 5r^3}{1 - 20r + 110rr - 200r^3 + 100r^4}$, which is reduced to $\frac{476}{723}$ by changing the Quantity r into its value $\frac{1}{16}$. Now subtracting the Fraction $\frac{476}{723}$ from Unity, the remainder will be the Fraction $\frac{247}{723}$, the Numerators of which two Fractions express the Odds of the First Plays Ending before the Second, which consequently will be as 476 to 247, or 27 to 14 nearly.

If in the foregoing Problem, the Skill of the Gamesters had been in any proportion of inequality, the Problem might have been Solved with the same ease.

When in a Problem of this nature the number of Stakes to be lost by either A or B , does not exceed the number Three, the Problem may be always readily Solved without the use of the Theorem inserted in our Third Lemma; tho' the number of Stakes between C or D be never so great. For which reason, if any one has the curiosity to try, if from the Solution of several Cases of this Problem, some Rule may not be discovered for Solving the same generally; it will be

convenient he should compare together the different Solutions, which may result from the supposition that the Stakes to be lost by either *A* or *B* are Two or Three; and then the Case of the foregoing Problem may also be compared with all the rest: Yet as these Trials might not perhaps be sufficient to discover any Analogy between those Solutions, I have thought fit to add a new Theorem in this place, whereby Four Cases more of this Problem may be Solved, *viz.* When the number of Stakes to be lost by *A* or *B*, and by *C* or *D*, are 4 and 6, 4 and 7, 5 and 6, 5 and 7: The Theorem being as follows.

If there be a Series of Terms whose Relation is expressed by the Index $l + m + n$, and there be likewise another Series of Terms whose Relation is expressed by the Index $p + q$; and the Corresponding Terms of those two Series be Multiplied together: Then the Index of the Relation in the Third Series, resulting from the Multiplication of their corresponding Terms, will be expressed by the Quantities.

$$\begin{aligned}
 &+ 2mq + lmpq + 2lnqq \\
 lp + llq + np^3 - mmqq - mnpqq + nnq^3. \\
 &+ mpp + 3npq + lnppq
 \end{aligned}$$

It is to be observed, that altho' these sorts of Theorems might be applicable to the finding of the Relation of those Terms, which are the Products of the corresponding Terms of two different Series, both of which consist of Terms whose last Differences are equal to nothing; yet there will be no necessity to use them for that purpose, that Relation being to be found much shorter, as follows.

Let *e* and *f* denote the rank of those Differences which are respectively equal to nothing in each Series; then the Quantity $e + f - 1$ will denote the rank of that Difference which is equal to nothing, in the Series resulting from the Multiplication of the corresponding Terms of the other two; and consequently the Relation of the Terms of this New Series will easily be obtained by our first Lemma.

After

After having given the Solution of several sorts of Problems, each of them containing some degree of Difficulty not to be met with in any of the rest; and having thereby laid a sufficient foundation for solving the most intricate cases that may occur in this Subject of Chances, it might almost seem superfluous to add any thing to this Tract: Yet considering that a Variety of Examples is the properest means of making Rules easy and familiar; and designing to be as useful as possible to those of my Readers, who perhaps may not be so well versed in Algebraical Calculations, I have chose to fill up the remaining Pages of this Book, with some easy Problems relating to the Games which are most in use, such as HAZARD, WHISK, PIQUET, &c. and to enlarge a little more upon the Doctrine of Combinations.

PROBLEM XLVII.

TO find at HAZARD the Advantage of the Setter upon all Suppositions of Main and Chance.

SOLUTION.

LET the whole Money Play'd for be considered as a common Stake, upon which both the Setter and Caster have their several Expectations; then let those Expectations be determined in the following manner.

First, Let it be supposed that the Main is *vii*; then if the Chance of the Caster be *vi* or *viii*, it is plain that the Setter having Six Chances to win and Five to lose, his Expectation will be $\frac{6}{11}$ of the Stake: But there being Ten Chances

out

out of Thirty-six for the Chance to be *vi* or *viii*, it follows, that the Expectation of the Setter, resulting from the Probability of the Chance being *vi* or *viii*, will be $\frac{10}{36}$ multiplied by $\frac{6}{11}$, or $\frac{60}{11}$ to be divided by 36.

Secondly, If the Main being *vii*, the Chance should Happen to be *v* or *ix*; then the Setter having Six Chances to win and Four to lose, his Expectation will be $\frac{6}{10}$ or $\frac{3}{5}$ of the Stake: But there being Eight Chances in Thirty-six for the Chance to be *v* or *ix*, it follows, that the Expectation of the Setter, resulting from the Probability of that Chance, will be $\frac{8}{36}$ multiplied by $\frac{3}{5}$, or $\frac{24}{5}$ to be divided by 36.

Thirdly, If the Main being *vii*, the Chance should Happen to be *iv* or *x*; then the Setter having Six Chances to win and Three to lose, his Expectation will be $\frac{6}{9}$ or $\frac{2}{3}$ of the Stake: But there being Six Chances out of Thirty-six for the Chance to be *iv* or *x*, it follows, that the Expectation of the Setter, resulting from the Probability of that Chance, will be $\frac{6}{36}$ multiplied by $\frac{2}{3}$, or 4 divided by 36.

Fourthly, If the Main being *vii*, the Caster should Happen to throw *ii*, *iii*, or *xii*; then the Expectation of the Setter will be the whole Stake, for which there being Four Chances in Thirty-six, it follows, that the Expectation of the Setter, resulting from the Probability of those Cases, will be $\frac{4}{36}$ of the Stake, or 4 divided by 36.

Lastly, If the Main being *vii*, the Caster should Happen to throw *vii* or *xi*, the Setter loses his Expectation.

From the Solution of the foregoing particular Cases it follows, that the Main being *vii*, the Expectation of the Setter will be exprest by the following Quantities, *viz.* $\frac{60}{11} + \frac{24}{5} + \frac{4}{1} + \frac{4}{1}$

which may be reduced to $\frac{251}{495}$. Now this Fraction being subtracted from Unity, to which the whole Stake is supposed equal, there will remain the Expectation of the Caster *viz.* $\frac{244}{495}$.

But the Probabilities of winning being always proportional to the Expectations, on supposition of the Stake being fixt, it follows, that the Probabilities of winning for the Setter and

and Caster are respectively Proportional to the two numbers 251 and 244, which properly denote the Odds of winning.

Now, if we suppose each Stake to be 1, or the whole Stake to be 2, the Gain of the Setter will be exprest by the Fraction $\frac{7}{495}$, it being the Difference of the Odds divided by their Sum, which supposing each Stake to be a Guinea, will be about $3d : 2\frac{1}{2}f$.

By the same Method of Process, it will be found that the Main being *vi* or *viii*, the Gain of the Setter will be $\frac{167}{7128}$, which is about $6d : \frac{1}{6}f$ in a Guinea.

It will also be found that the Main being *v* or *ix*, the Gain of the Setter will be $\frac{43}{2835}$, which is about $4d : 2\frac{1}{9}f$ in a Guinea.

Coroll. 1. If each particular Gain made by the Setter, in the Case of any Main, be respectively Multiplied by the number of Chances there are for that Main to come up, and the Sum of the Products be divided by the number of all those Chances, the Quotient will exprest the Gain of the Setter before a Main is thrown: from whence it follows, that the Gain of the Setter, if he be resolved to set upon the first Main, may be estimated to be $\frac{344}{2835} + \frac{1670}{7128} + \frac{42}{495}$ to be divided by 24; which being reduced will be $\frac{2}{109}$ very nearly, or about $4d : 2\frac{1}{10}f$.

Coroll. 2. The Probability of no Main is to the Probability of a Main, as $109 + 2$ to $109 - 2$, or as 111 to 107.

Coroll. 3. The Loss of the Caster's hand, if each throw be for a Guinea, and he confine himself to hold it as long as he wins, will be $\frac{4}{111}$ or about $9d$. in all, the Demonstration of which may be deduced from our XXIVth Problem.

PROBLEM XLVIII.

IF Four Gamesters play at WHISK; What are the Odds that any two of the Partners that are pitch'd upon, have not the Four Honours?

SI

SOLU.

SOLUTION.

First, suppose those two Partners to have the Deal, and the last Card which is turn'd up to be an Honour.

From the supposition of these two Cases, we are only to find what Probability the Dealers have of taking Three set Cards in Twenty five, out of a Stock containing Fifty one. To resolve this the shortest way, recourse must be had to the Theorem given in the *Corollary* of our XXth Problem, in which making the Quantities n, c, d, p, a , respectively equal to the numbers 51, 25, 26, 3, 3, the Probability required will be found to be $\frac{25 \times 24 \times 23}{51 \times 50 \times 49}$ or $\frac{92}{833}$.

Secondly, If the Card which is turn'd up be not an Honour, then we are to find what Probability the Dealers have, of taking Four given Cards in Twenty five out of a Stock containing Fifty one, which by the aforesaid Theorem will be found to be $\frac{25 \times 24 \times 23 \times 22}{51 \times 50 \times 49 \times 48}$, or $\frac{253}{4998}$.

But the Probability of taking the Four Honours being to be estimated before the last Card is turn'd up; and there being Sixteen Chances in Fifty two, or Four in Thirteen for an Honour to turn up, and Nine in Thirteen against it; it follows, that the Fraction expressing the Probability of the First Case ought to be Multiplied by 4; that the Fraction expressing the Probability of the Second ought to be Multiplied by 9; and that the sum of those Products ought to be divided by 13; which being done, the Quotient $\frac{115}{1666}$, or $\frac{2}{29}$ nearly, will express the Probability required.

Corollary, By the help of the abovesaid Theorem, the following Conclusions may easily be verified.

It is 27 to 2 nearly that the two Dealers have not the Four Honours.

It is 23 to 1 nearly that the two Eldest have not the Four Honours.

It is 8 to 1 nearly that neither one Side nor the other have the Four Honours.

It is 13 to 7 nearly that the two Dealers do not reckon Honours.

It is 20 to 7 nearly that the two Eldest do not reckon Honours.

It is 25 to 16 nearly that either one Side or the other do reckon Honours, or that the Honours are not equally divided.

PROBLEM XLIX. Of RAFFLING.

IF any number of Gamesters *A, B, C, D &c.* Play at Raffles:
What is the Probability that the first of them having got his Chance wins the Money of the Play?

SOLUTION.

IN order to Solve this Problem, it is necessary to have a Table ready compos'd, of all the Chances which there are in three Raffles, which Table is the following. Wherein

The first Column contains the number of Points which are supposed to have been thrown by *A* in three Raffles.

The second Column contains the number of Chances which *A* has to win if his Points be above *xxxi*, or the number of Chances he has to lose if they be either *xxxi* or below it.

The third Column contains the number of Chances which *A* has to lose, if his Points be above *xxxi*, or to win if they be either *xxxi* or below it.

The Fourth Column contains the number of Chances which he has for an equality of Chance.

The Construction of this Table easily flows from the consideration of the number of Chances which there are in a single Raffle; whereof *xviii* or *iii*, have 1 Chance; *xvii* or *iv*, 3 Chances; *xvi* or *v*, 6 Chances; *xv* or *vi*, 4 Chances; *xiv* or *vii*, 9 Chances; *xiii* or *viii*, 9 Chances; *xii* or *ix*, 7 Chances; *xi* or *x*, 9 Chances; which number of Chances being duly Combined will afford all the Chances of Three Raffles.

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A TABLE of all the CHANCES which are in three Raffles.

Points		Chances to win or lose.	Chances to win or lose.	Equality of Chance.
<i>liv</i>	<i>ix</i>	884735	0	1
<i>liii</i>	<i>x</i>	884726	1	9
<i>lii</i>	<i>xi</i>	884681	10	45
<i>li</i>	<i>xii</i>	884534	55	147
<i>l</i>	<i>xiii</i>	884165	202	369
<i>xlxix</i>	<i>xiv</i>	883400	571	765
<i>xlvi</i>	<i>xv</i>	881954	1336	1446
<i>xlvi</i>	<i>xvi</i>	879470	2782	2484
<i>xlvi</i>	<i>xvii</i>	875501	5266	3969
<i>xlvi</i>	<i>xviii</i>	869632	9235	5869
<i>xliv</i>	<i>xix</i>	861199	15104	8433
<i>xlvi</i> < Or >	<i>xx</i>	849706	23537	11493
<i>xlvi</i>	<i>xxi</i>	834679	35030	15027
<i>xli</i>	<i>xxii</i>	815392	50057	19287
<i>xl</i>	<i>xxiii</i>	791506	69344	23886
<i>xxxix</i>	<i>xxiv</i>	762838	93230	28668
<i>xxxviii</i>	<i>xxv</i>	728971	121898	33867
<i>xxxvii</i>	<i>xxvi</i>	690100	155765	38871
<i>xxxvi</i>	<i>xxvii</i>	646929	194636	43171
<i>xxxv</i>	<i>xxviii</i>	599472	237807	47457
<i>xxxiv</i>	<i>xxix</i>	548865	285264	50607
<i>xxxiii</i>	<i>xxx</i>	496314	335871	52551
<i>xxxii</i>	<i>xxxi</i>	442368	388422	53946
			Sum	442368
				442368
				884736

This

This being once supposed, let it be required to find the Probability which *A* has of winning, when the number of his Points being $x/$, there is but one Gamester *B* besides himself.

Take the number 791506, which in the second Column stands over against the number $x/$, to be found in the First. Take also one half of the number which in the Fourth Column stands over against the said Number $x/$, which half is 11943. Let these two Numbers viz. 791506 and 11943 be added together, and their Sum 803449 being divided by 884736, which is the Number of all the Chances, the Quotient, viz. $\frac{803449}{884736}$ will express the Probability required.

Now this Fraction being Subtracted from Unity, and the remainder being $\frac{81287}{884736}$, it follows that the Numerators of these two Fractions, viz. 803449 and 81287 do express the Odds of winning, which may be reduc'd to 89 and 9 nearly.

But if the Number of Points which *A* has thrown for his Chance being $x/$ as above, there be two other Gamesters *B* and *C* besides himself, the Probability which he has of winning will be found thus.

Take the Square of the Number set down over against $x/$ in the second Column, which Square is 626481748036. Take also the Product of that Number by the Number set down over against $x/$ in the Fourth Column, which Product is 18905912316. Lastly, take the third part of the Square of the Number set down in the Third Column, which third part will be 190180332, and let all those numbers be added together: Then their Sum being divided by the Square of the whole Number of Chances, viz. by 782757789696, the Quotient $\frac{645577840684}{782757789696}$ will express the Probability required; from whence it may be concluded that the Odds of winning are nearly as 33 to 7.

N. B. If some of the last figures in the Numbers of the foregoing Table be neglected, the Operation will be shortened, and a sufficient Approximation obtain'd by help of the remaining Figures.

From what we have said it follows, that *A* having $x/$ for the number of his Points, has less advantage when he Plays

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against

against One than when he plays against Two: For supposing each Man's Stake be a Guinea, he has in the first Case 89 Chances for winning 1, and 9 Chances for losing 1:

From whence it follows that his Gain is $\frac{89-9}{98}$ or $\frac{80}{98}$ which is about 17 *sh.* 6 *d.*

But in the second Case, supposing also each Man's Stake to be a Guinea, he has 33 Chances for winning 2, and 7 Chances for losing 1:

Whence it appears, that his Gain in this Case is $\frac{2 \times 33 - 7}{40}$ or $\frac{59}{40}$ which is about 1 *l.* — 12 *sh.* — 8 *d.* But Note, that it is not to be concluded from this single Instance, that the Gain of *A* will always increase with the number of Gamesters.

If the number of Gamesters be never so many, let *p* be their number, let *a* be the number of Chances which *A* has for winning when he has thrown his Chance, let *m* be the number of Chances which there are for an Equality of Chance between *A* and any of the other Gamesters; Lastly, let the whole number of Chances be denoted by *S*: Then the Probability which *A* has of winning will be expressed by the following Series.

$$\frac{a^{p-1} + \frac{p-1}{2} m a^{p-2} + \frac{p-1}{2} \times \frac{p-2}{3} m m a^{p-3} + \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4} m^3 a^{p-4} \&c.}{S^{p-1}}$$

which Series is composed of the Terms of the Binomial $(a + m)^{p-1}$ reduced into a Series, all its Terms being divided by 1, 2, 3, 4, 5 &c. respectively.

The foregoing Theorem may be useful, not only for solving any Case of the present Problem, but also an infinite Variety of other Cases, in those Games wherein there is no Advantage in the order of Play: And the Application of it to Numbers will be found easy, to those who understand how to use Logarithms.

PROBLEM L.

TO find what Probability there is, that any Number of Cards of each Suit may be contained in a given number of them taken out of a given Stock.

SOLUTION.

First, Find the whole number of Chances there are for taking the given number of Cards out of the given Stock.

Secondly, Find all the particular Chances there are for taking each given number of Cards of each Suit out of the whole number of Cards belonging to that Suit.

Thirdly, Multiply all those particular Chances together; then divide the Product by the whole number of Chances, and the Quotient will express the Probability required.

Thus, If it be proposed to find the Probability of taking Four Hearts, Three Diamonds, Two Spades and One Club, in Ten Cards taken out of a Stock containing Thirty-two.

Find the whole number of Chances for taking ten Cards out of a Stock containing two and Thirty; which is properly Combining two and Thirty Cards Ten and Ten. To do this, write down all the Numbers from 32 inclusively to 22 exclusively, so as to have as many Terms as there are Cards to be Combined; then write under each of them respectively all the numbers from One to Ten inclusively, thus,

$$\begin{array}{r} 32 \times 31 \times 30 \times 29 \times 28 \times 27 \times 26 \times 25 \times 24 \times 23 \\ \hline 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \end{array}$$

Let all the numbers of the upper Row be Multiplied together; let also all the numbers of the lower Row be Multiplied together, then the first Product being divided by the second, the Quotient will express the whole number of Chances required, which will be 64512240.

By the like Operation the number of Chances for taking Four Hearts out of Eight, will be found to be

$$\frac{8 \times 7 \times 6 \times 5}{1 \times 2 \times 3 \times 4} = 70$$

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The number of Chances for taking Three Diamonds out of Eight will also be found to be $\frac{8 \times 7 \times 6}{1 \times 2 \times 3} = 56$.

The number of Chances for taking Two Spades out of Eight will in the same manner be found to be $\frac{8 \times 7}{1 \times 2} = 28$.

Lastly, The number of Chances for taking One Club out of Eight will be found to be $\frac{8}{1} = 8$.

Wherefore, Multiplying all these particular Chances together *viz.* 70, 56, 28, 8, the Product will be 878080; which being Divided by the whole number of Chances, the Quotient $\frac{878080}{64512240}$, or $\frac{2}{101}$ nearly, will express the Probability required: From whence it follows, that the Odds against taking Four Hearts, Three Diamonds, Two Spades and One Club in Ten Cards, are very near 99 to 2.

It is to be observed that the Operations whereby the Number of Chances is determined, may always be contracted, except in the single case of taking one Card only of a given Suit. Thus, If it were proposed to shorten the Fraction

$\frac{32 \times 31 \times 30 \times 29 \times 28 \times 27 \times 26 \times 25 \times 24 \times 23}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10}$, which determines the number of all the Chances belonging to the foregoing Problem: Let it be considered whether the Product of any two or more Terms of the Denominator, being Multiplied together, be equal to any one of the Terms of the Numerator; if so, all those Terms may be expunged out of both Denominator and Numerator. Thus the Product of the three Numbers 2, 3, 4, which are in the Denominator, being equal to the Number 24, which is in the Numerator, it follows, that the three Numbers 2, 3, 4 may be expunged out of the Denominator, and at the same time the Number 24 out of the Numerator. For the same reason the Numbers 5 and 6 may be expunged out of the Denominator, and the Number 30 out of the Numerator, which will reduce the Fraction to be

$$\frac{32 \times 31 \times 30 \times 29 \times 28 \times 27 \times 26 \times 25 \times 24 \times 23}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10}$$

It ought likewise to be considered whether there be any of the remaining Numbers in the Denominator that Divide exactly any of the remaining Numbers of the Numerator.

If

If so, those Numbers are to be expunged out of the Denominator and Numerator, but the respective Quotients of the Terms of the Numerator divided by those of the Denominator, are to be substituted in the room of those Terms of the Numerator. Thus the Terms 7, 8, and 9 of the Denominator dividing exactly the Numbers 28, 32 and 27 of the Numerator, and the Quotients being 4, 4 and 3 respectively, all the Numbers 7, 8, 9, 28, 32, 27 ought to be expunged, and the Quotients 4, 4, and 3 substituted in the room of 28, 32, 27 respectively, in the following manner;

$$\frac{32 \times 31 \times 30 \times 29 \times 28 \times 27 \times 26 \times 25 \times 24 \times 23}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10}$$

It ought also to be considered, whether the remaining Terms of the Denominator have any common Divisor with any of the remaining Terms of the Numerator; if so, dividing those Terms by their common Divisors, the respective Quotients ought to be substituted in the room of the Terms of the Numerator. Thus, the only remaining Term in the Denominator, besides Unity, being 10, which has a common Divisor with one of the remaining Terms of the Numerator, viz. 25, and that common Divisor being 5, let 10 and 25 be respectively divided by the common Divisor 5, and let the respective Quotients 2 and 5 be substituted in the room of them, and the Fraction will be reduced to the following, viz.

$$\frac{32 \times 31 \times 30 \times 29 \times 28 \times 27 \times 26 \times 25 \times 24 \times 23}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10}$$

Lastly, Let the remaining Number 2 in the Denominator divide any of the Numbers of the Numerator which are divisible by it, such as 26, and let those two Numbers be expunged; but let the Quotient of 26 by 2, viz. 13, be substituted in the room of 26: And then the Fraction, neglecting unity, which is the only Term remaining, may be reduced to $4 \times 31 \times 29 \times 4 \times 3 \times 13 \times 5 \times 23$, the Product of which Numbers is 64512240, as we have found it before.

The foregoing Solution being well understood, it will be easy to enlarge the Problem, and to find the Probability of

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taking

taking at least Four Hearts, Three Diamonds, Two Spades, and One Club, in Eleven Cards; the finding of which depends upon the four following Cases, *viz.* taking

- 5 Hearts, 3 Diamonds, 2 Spades, 1 Club,
- 4 Hearts, 4 Diamonds, 2 Spades, 1 Club,
- 4 Hearts, 3 Diamonds, 3 Spades, 1 Club,
- 4 Hearts, 3 Diamonds, 2 Spades, 2 Clubs.

Now the number of Chances for the First Case will be found to be 702464, for the Second 1097600, for the Third 1756160, for the Fourth 3073280; which Chances being added together, and their sum divided by the whole number of Chances for taking Eleven Cards out of Thirty two, the Quotient will be $\frac{6628504}{12902480}$, which may be reduced to $\frac{5}{97}$ nearly.

From whence it may be concluded, that the Odds against the taking of Four Hearts, Three Diamonds, Two Spades, and One Club, in Eleven Cards, that is, so many at least of every sort, is about 92 to 5.

And by the same Method it would be easy to solve any other Case of the like nature, let the number of Cards be what it will.

PROBLEM LI.

TO find at PIQUET the Probability which the Dealer has for taking One Ace or more in Three Cards, having none in his Hands.

SOLUTION.

FROM the number of all the Cards, which are Thirty two, subtracting Twelve which are in the Dealers Hands, there remains Twenty, among which are the Four Aces.

From whence it follows, that the number of all the Chances for taking any three Cards in the Bottom, are the number of Combinations which Twenty Cards may afford, being taken Three and Three; which, by the Rule given in the preceding Problem, will be found to be

$$\frac{20 \times 19 \times 18}{1 \times 2 \times 3} \text{ or } 1140.$$

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The number of all the Chances being thus obtained, find the number of Chances for taking one Ace precisely, with two other Cards; find next the number of Chances for taking Two Aces precisely with any other Card; Lastly, find the number of Chances for taking Three Aces: Then these Chances being added together, and their sum divided by the whole number of Chances, the Quotient will express the Probability required.

But by the Directions given in the preceding Problem, it appears, that the number of Chances for taking One Ace precisely are $\frac{4}{1}$ or 4; and that the number of Chances for taking any two other Cards are $\frac{16 \times 15}{1 \times 2}$ or 120: From whence it follows, that the number of Chances for taking One Ace precisely with any two other Cards, is equal to 4×120 or 480.

In like manner it appears, that the number of Chances for taking Two Aces precisely is equal to $\frac{4 \times 3}{1 \times 2}$ or 6, and that the number of Chances for taking any other Card is $\frac{16}{1}$ or 16; from whence it follows, that the number of Chances for taking Two Aces precisely with any other Card is 6×16 or 96.

Lastly, It appears that the Number of Chances for taking Three Aces is equal to $\frac{4 \times 3 \times 2}{1 \times 2 \times 3}$ or 4.

Wherefore the Probability required will be found to be $\frac{480 + 96 + 4}{1140}$ or $\frac{580}{1140}$; which Fraction being subtracted from Unity, the remainder, viz. $\frac{560}{1140}$ will express the Probability of not taking an Ace in Three Cards: From whence it follows, that it is 580 to 560, or 29 to 28, that the Dealer takes One Ace or more in three Cards.

The preceding Solution may be very much contracted, by inquiring at first what the Probability is of not taking an Ace in Three Cards, which may be done thus:

The number of Cards in which the Four Aces are contained being Twenty, and consequently the number of Cards out of which the Four Aces are excluded being Sixteen, it follows, that the number of Chances which there are for the taking Three Cards, among which no Ace shall be found,

is the number of Combinations which Sixteen Cards may afford, being taken Three and Three; which number of Combinations by the preceding Problem will be found to be

$$\frac{16 \times 15 \times 14}{1 \times 2 \times 3} \text{ or } 560.$$

But the number of all the Chances which there are for taking any Three Cards in Twenty, has been found to be 1140; from whence it follows, that the Probability of not taking an Ace in Three Cards is $\frac{560}{1140}$; and consequently that the Probability of taking One or more Aces in Three Cards is $\frac{580}{1140}$: The same as before.

In the like manner, if we would find the Probability which the Eldest has of taking One Ace or more in his Five Cards, he having none in his Hands; the severall Chances may be calculated as follows.

First, The number of Chances for taking One Ace and Four other Cards will be found to be 7280.

Secondly, The number of Chances for taking Two Aces and Three other Cards will be found to be 3360.

Thirdly, The number of Chances for taking Three Aces and two other Cards will be found to be 480.

Fourthly, The number of Chances for taking Four Aces and any other Card will be found to be 16.

Lastly, The number of Chances for taking any Five Cards will be found to be 15504.

Let the sum of all the particular Chances, viz. 7280 + 3360 + 480 + 16 or 11136, be divided by the sum of all the Chances, viz. by 15504, and the Quotient $\frac{11136}{15504}$ will express the Probability required.

Now the foregoing Fraction being subtracted from Unity, the remainder, viz. $\frac{4368}{15504}$ will express the Probability of not taking an Ace in Five Cards; wherefore the Odds of taking an Ace in Five Cards are 11136 to 4368, or 5 to 2 nearly.

But if the Probability of not taking an Ace in Five Cards be at first inquired into, the Work will be very much shortened; for it will be found to be $\frac{16 \times 15 \times 14 \times 13 \times 12}{1 \times 2 \times 3 \times 4 \times 5}$ or 4368, to be divided by the whole number of Chances, viz. by 15504, which makes it as before, equal to $\frac{4368}{15504}$.

But

But suppose it were required to find the Probability which the Eldest has of taking an Ace and a King in Five Cards, he having none in his Hands: Let the following Chances be found, *Viz.*

- 1 For One Ace, One King and Three other Cards.
- 2 For One Ace, Two Kings and Two other Cards.
- 3 For One Ace, Three Kings and any other Card.
- 4 For One Ace and Four Kings.
- 5 For Two Aces, One King and Two other Cards.
- 6 For Two Aces, Two Kings and any other Card.
- 7 For Two Aces and Three Kings.
- 8 For Three Aces, One King and any other Card.
- 9 For Three Aces and Two Kings.
- 10 For Four Aces and One King.
- 11 For taking any Five Cards in Twenty.

Among these Cases, there being four Pairs that are alike, *viz.* the Second and Fifth, the Third and Eighth, the Fourth and Tenth, the Seventh and Ninth; it follows, that there are only Seven Cases to be Calculated, whereof the First, Sixth and Eleventh, are to be taken singly; but the Second, Third, Fourth and Seventh, to be doubled. Now the Operation is as follows.

The *First* Case has $\frac{4}{1} \times \frac{4}{1} \times \frac{12 \times 11 \times 10}{1 \times 2 \times 3}$ or 3520 Chances.

The *Second*, $\frac{4}{1} \times \frac{4 \times 3}{1 \times 2} \times \frac{12 \times 11}{1 \times 2}$ or 1584, the double of which is 3168 Chances.

The *Third*, $\frac{4}{1} \times \frac{4 \times 3 \times 2}{1 \times 2 \times 3} \times \frac{12}{1}$ or 192, the double of which is 384 Chances.

The *Fourth*, $\frac{4}{1} \times \frac{4 \times 3 \times 2 \times 1}{1 \times 2 \times 3 \times 4}$ or 4, the double of which is 8 Chances.

The *Sixth*, $\frac{4 \times 3}{1 \times 2} \times \frac{4 \times 3}{1 \times 2} \times \frac{12}{1}$ or 432 Chances.

The *Seventh*, $\frac{4 \times 3}{1 \times 2} \times \frac{4 \times 3 \times 2}{1 \times 2 \times 3}$ or 24, the double of which is 48 Chances.

The *Eleventh*, $\frac{20 \times 19 \times 18 \times 17 \times 16}{1 \times 2 \times 3 \times 4 \times 5}$ or 15504, being the number of all the Chances for taking any Five Cards out of Twenty.

From whence it follows, that the Probability which the Eldest has for taking an Ace and a King in Five Cards, he having none in his Hands, will be exprest by the Fraction

$$\frac{3520 + 3168 + 384 + 8 + 432 + 48}{15504} \text{ or } \frac{7560}{15504}.$$

Let this Fraction be subtracted from Unity, and the remainder being $\frac{7944}{15504}$, the Numerators of these two Fractions, viz. 7560 and 7944, will exprest the proportion of Probability that there is, of taking or not taking an Ace and a King in Five Cards; which two numbers may be reduced nearly to the proportion of 20 to 21.

By the same Method of Process, any Case relating to WHISK might be Calculated, tho' not so expeditiously, as by the Method explained in the *Corollary* of our XXth Problem: For which reason the Reader is desired to have recourse to the Method therein explained, when any other Case of the like nature happens to be proposed.

PROBLEM LI.

TO find the Probability of taking any number of Suits, in a given number of Cards taken out of a given Stock; without specifying what number of Cards of each Suit shall be taken.

SOLUTION.

Suppose the number of Cards to be taken out of the given Stock to be Eight, the number of Suits to be Four, and the number of Cards in the Stock to be Thirty-two.

Let all the Variations that may happen, in taking One Card at least of each Suit, be written down in order, as follows,

1,	1,	1,	5,
1,	1,	2,	4,
1,	1,	3,	3,
1,	2,	2,	3,
2,	2,	2,	2.

Then

Then supposing any particular Suits to be appropriated at pleasure to the Numbers belonging to the First Case, as if it were required, for Instance, to take One Heart, One Diamond, One Spade and Five Clubs; let the Probability of the same be inquired into, which, by our *Lth* Problem, will be found to be $\frac{28672}{10518300}$; but the Problem not requiring the Suits to be confined to any number of Cards of each Sort, it follows, that this Probability ought to be increased in proportion to the number of Permutations, or Changes of Order, which Four Things may undergo, whereof Three are alike. Now this number of Permutations is Four, and consequently the Probability of the First Case, that is, of taking Three Cards of three different Suits, and five Cards of a Fourth Suit, in Eight Cards, will be the Fraction $\frac{28672}{10518300}$ multiplied by 4, or $\frac{114688}{10518300}$.

In the same manner the Probability of the Second Case, supposing it were confined to One Heart, One Diamond, Two Spades, and Four Clubs, would be found to be $\frac{125440}{10518300}$, which being multiplied by 12, *viz.* by the number of Permutations which Four Sorts may undergo, whereof Two are alike, and the other Two differing, it will follow, that the Probability of the Second Case, taken without any restriction, will be expressed by the Fraction $\frac{1505280}{10518300}$.

The Probability of the Third Case will likewise be found to be $\frac{1204224}{10518300}$.

The Probability of the Fourth will be found to be $\frac{4214784}{10518300}$.

Lastly, The Probability of the Fifth will be found to be $\frac{614656}{10518300}$. These Fractions being added together, their sum, *viz.* $\frac{7653632}{10518300}$, will express the Probability of taking the Four Suits in Eight Cards.

Let this last Fraction be subtracted from Unity, and the remainder being $\frac{2864668}{10518300}$, it follows that 'tis the Odds of 7653632 to 2864668, or 8 to 3 nearly, that the Four Suits may be taken in Eight Cards, out of a Stock containing Thirty-two.

The only difficulty remaining in this matter, is the finding readily the number of Permutations which any number of

of Things may undergo, when either they be all different, or when some of them be alike. The Solution of which may be deduced from what we have said in the *Corollary* of our *XVIIth* Problem, and may be explained as follows, in words at length.

Let all the numbers that are from Unity to that number which expresses how many Things are to be Permuted, be written down in order; Multiply all those Numbers together, and the Product of them all will express the number of their Permutations, if they be all different. Thus the number of Permutations which Ten things are capable of, is the Product of all the Numbers $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10$, which is equal to 3628800.

But if some of them be alike, as suppose Four of One sort, Three of another, Two of a Third, and One of a Fourth, write down as before all the Numbers $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10$; then write under them as many of those Numbers as there are Things of the First sort that are alike, which in this Case being Four, write the Numbers $1 \times 2 \times 3 \times 4$, beginning at Unity, and following in order. Write also as many of those Numbers as there are Things of the second sort that are alike, *viz.* $1 \times 2 \times 3$, still beginning at Unity. In the same manner write as many more as there are Things of the Third sort that are alike, *viz.* 1×2 ; and so on: Which being represented by the Fraction

$$\frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10}{1 \times 2 \times 3 \times 4 \times 1 \times 2 \times 3 \times 1 \times 2 \times 1},$$

let all the numbers of the upper Row be Multiplied together, let also all the numbers of the lower Row be Multiplied together, and the First Product being divided by the Second, the Quotient 12600 will express the number of Permutations required.

By this Method of Permutations, the Probability of throwing any determinate number of Faces of the like sort, with any given number of Dice, may easily be found. Thus, suppose it were required to find the Probability of throwing an Ace, a Two, a Three, a Four, a Five, and a Six with six Dice. It is plain that there are as many Chances for doing it, as there are Changes or Permutations in the Order

Order or Place of six different Things, suppose of the Six Letters *a, b, c, d, e, f*, which by the Rule above given would be 720, *viz.* the Product of the numbers 1, 2, 3, 4, 5, 6: For tho' the Dice are not considered as changing their Places, or as affording any Variation upon the score of the different Situation they may have in Respect to one another, being thrown upon a Table; yet they ought to be considered as changing their Faces, which is equivalent to their changing of Place. Now the number of all the Chances upon Six Dice, being the number 6 Multiplied into it self, as many times wanting one as there are Dice, *viz.* $6 \times 6 \times 6 \times 6 \times 6 \times 6$ or 46656, it follows, that the Probability required will be exprest by the Fraction $\frac{720}{46656}$, and consequently, that the Odds against throwing the Faces undertaken, will be 46656—720 to 720, or 64 to 1 nearly.

In the same manner suppose it were required to find the Probability of throwing One Ace, Two Two's and Three Three's with Six Dice. The number of Chances for the doing it being equal to the number of Permutations which there are in the six Letters *abbccc*, it follows, by the Rule before delivered, that the number of those Chances will be 60, *viz.* the Fraction $\frac{1 \times 2 \times 3 \times 4 \times 5 \times 6}{1 \times 1 \times 2 \times 1 \times 2 \times 3}$; and consequently that the Probability required will be $\frac{60}{46656}$, and the Odds against the doing it 46656—60 to 60, or 776 to 1 nearly.

If it were required to find the Probability of throwing Two Aces, Two Two's and Two Three's with 6 Dice, the number of Chances for doing it being $\frac{1 \times 2 \times 3 \times 4 \times 5 \times 6}{1 \times 2 \times 1 \times 2 \times 1 \times 2}$ or 90, and the number of all the Chances upon Six Dice being 46656, it follows, that the Probability required will be exprest by the Fraction $\frac{90}{46656}$.

Again, if it were required to find the Probability of throwing Three Aces and Three Sixes, the number of Chances for doing it being $\frac{1 \times 2 \times 3 \times 4 \times 5 \times 6}{1 \times 2 \times 3 \times 1 \times 2 \times 3}$ or 20, and the number of all the Chances 46656, the Probability required will be exprest by the Fraction $\frac{20}{46656}$.

PROBLEM LIII.

TO find at HAZARD the Chance of the Caster, when the Main being given, he Throws to any given number of Points.

SOLUTION.

THis being easily reduced to our XLVIIth Problem, it is thought sufficient to exhibit the Solution of its different Cases in the following Table, which shews the Odds for or against the Caster.

Points Thrown to	MAIN V.		
		exactly	nearly
i	Against the Caster	538 to 407	or 37 to 28.
ii	For the Caster —	989 to 901	or 45 to 41.
iii	For the Caster —	2293 to 1487	or 37 to 24.
iv	For the Caster —	2293 to 1487	or 37 to 24.
v	Against the Caster	2117 to 1663	or 14 to 11.
vi.	Against the Caster	2467 to 1313	or 62 to 33.
	MAIN VI.		
i	Against the Caster	2879 to 1873	or 83 to 54.
ii	Against the Caster	2483 to 2269	or 58 to 53.
iii	For the Caster —	2621 to 2131	or 16 to 13.
iv	For the Caster —	2621 to 2131	or 16 to 13.
v	Against the Caster	2483 to 2269	or 58 to 53.
vi.	Against the Caster	2483 to 2269	or 58 to 53.
	MAIN VII.		
i	Against the Caster	629 to 361	or 7 to 4.
ii	Against the Caster	277 to 218	or 14 to 11.
iii	For the Caster —	251 to 244	or 36 to 35.
iv	For the Caster —	251 to 244	or 36 to 35.
v	For the Caster —	601 to 389	or 20 to 13.
vi.	For the Caster —	263 to 232	or 17 to 15.

MAIN

Points
Thrown
to

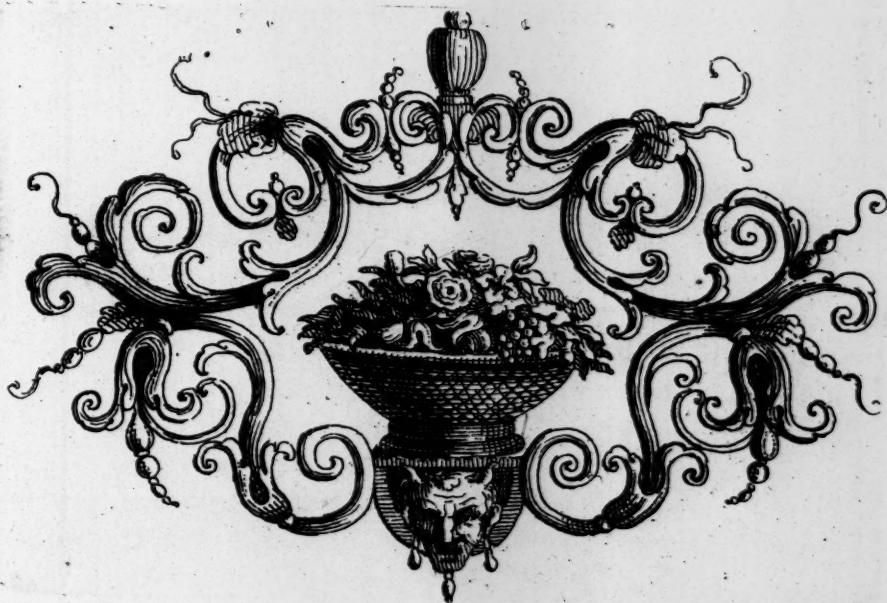
MAIN VIII.

i	Against the Caster 3275 to 1477 or 51 to 23.
ii	Against the Caster 2483 to 2269 or 58 to 53.
iii	For the Caster — 2621 to 2131 or 16 to 13.
iv	For the Caster — 2621 to 2131 or 16 to 13.
v	For the Caster — 2483 to 2269 or 58 to 53.
vi.	For the Caster — 2665 to 2087 or 83 to 65.

MAIN IX.

i	Against the Caster 2467 to 1313 or 62 to 33.
ii	Against the Caster 2117 to 1663 or 14 to 11.
iii	For the Caster — 2293 to 1487 or 37 to 24.
iv	For the Caster — 2293 to 1487 or 37 to 24.
v	For the Caster — 989 to 901 or 45 to 41.
vi.	Against the Caster 538 to 407 or 37 to 28.

F I N I S.



THE DOCTRINE OF CHANCES

MAIN VOLUME

1. The Doctrine of Chances — 10 00
2. The Doctrine of Chances — 10 00
3. The Doctrine of Chances — 10 00
4. The Doctrine of Chances — 10 00
5. The Doctrine of Chances — 10 00
6. The Doctrine of Chances — 10 00
7. The Doctrine of Chances — 10 00
8. The Doctrine of Chances — 10 00
9. The Doctrine of Chances — 10 00
10. The Doctrine of Chances — 10 00



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